Math 252 – Test 2: Version A

March 6, 2020 **Name:** Solution Set

Instructor: Patricia Wrean

Total: 25 points

1. (4 points) Find a second solution to the following DE, given that $y_1 = \cos(x^2)$ is a solution. You may assume that $x > 0$.

$$xy'' - y' + 4x^3y = 0$$

$$y'' - \frac{1}{x}y' + 4x^2y = 0$$

reduction of order: $P(x) = -\frac{1}{x}$

$$e^{-\int P(x)dx} = e^{\int \frac{1}{x}dx} = e^{\ln|x|} = x$$

$$y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{y_1^2} dx$$

$$= \cos(x^2) \int \frac{x}{\cos^2(x^2)} dx$$

$$= \cos(x^2) \int x \sec^2(x^2)\, dx$$

$$= \cos(x^2) \cdot \frac{\tan(x^2)}{2}$$

$$\boxed{y_2 = \frac{1}{2}\sin(x^2)} \text{ or just } \boxed{\sin(x^2)}$$

let $u = x^2$

$du = 2x\,dx$

$$\int x \sec^2(x^2) dx = \frac{1}{2}\int \sec^2 u\, du = \frac{1}{2}\tan u = \frac{1}{2}\tan x^2$$

2. (4 points) Consider the following DE.

$$x^2y'' + 5x\,y' + cy = 0$$

Solve this DE for the following values of c.

(a) $c = 4$

(b) $c = 5$

a) $x^2y'' + 5xy' + 4y = 0$

$m(m-1) + 5m + 4 = 0$

$m^2 + 4m + 4 = 0$

$(m+2)^2 = 0$

$m = -2, -2$

$$\boxed{y = C_1 x^{-2} + C_2 x^{-2} \ln x}$$

b) $x^2y'' + 5xy' + 5y = 0$

$m(m-1) + 5m + 5 = 0$

$m^2 + 4m + 5 = 0$

$$m = \frac{-4 \pm \sqrt{16 - 20}}{2}$$

$$= \frac{-4 \pm 2i}{2} = -2 \pm i$$

$$\boxed{y = x^{-2}\left[C_1 \cos(\ln x) + C_2 \sin(\ln x)\right]}$$

3. (6 points) Use the method of variation of parameters to solve the following DE.

$$y'' + y = \sec x$$

$\boxed{y_c}$ aux eqn: $m^2 + 1 = 0$

$m = \pm i$

$y_c = C_1 \cos x + C_2 \sin x$

$\boxed{y_p}$ variation of parameters with $y_1 = \cos x$

$y_2 = \sin x$

$f(x) = \sec x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

$$W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix} = \begin{vmatrix} 0 & \sin x \\ \sec x & \cos x \end{vmatrix} = -\sin x \sec x = -\tan x$$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix} = \begin{vmatrix} \cos x & 0 \\ -\sin x & \sec x \end{vmatrix} = \cos x \sec x = 1$$

$$u_1 = \int \frac{W_1}{W} \, dx = \int -\tan x \, dx = -\ln|\sec x|$$

$$u_2 = \int \frac{W_2}{W} \, dx = \int dx = x$$

$$y_p = u_1 y_1 + u_2 y_2$$
$$= (\ln|\sec x|) \cos x + x \sin x$$

$$\boxed{y = y_c + y_p = C_1 \cos x + C_2 \sin x - \cos x \ln|\sec x| + x \sin x}$$

4. (6 points) Consider the following differential equations and graphs.

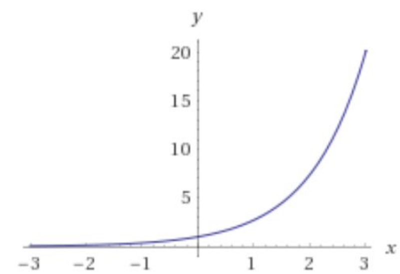
   (a) State the form of the particular solution y_p for the following. Leave your answer with undetermined coefficients. Please note that the complementary solution for the homogeneous equation is $y_c = C_1 e^{-2x} + C_2 e^{-4x}$.

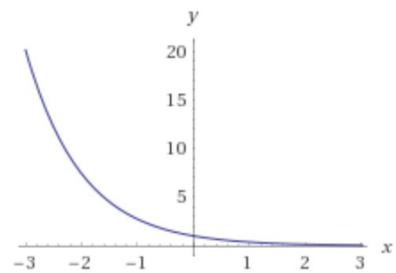
   (i) $y'' + 6y' + 8y = 5e^{-2x}$ $\qquad$ $\underline{Axe^{-2x}}$

   (ii) $y'' + 6y' + 8y = e^{x} + \cos(2x)$ $\qquad$ $\underline{Ae^{x} + B\cos 2x + C\sin 2x}$

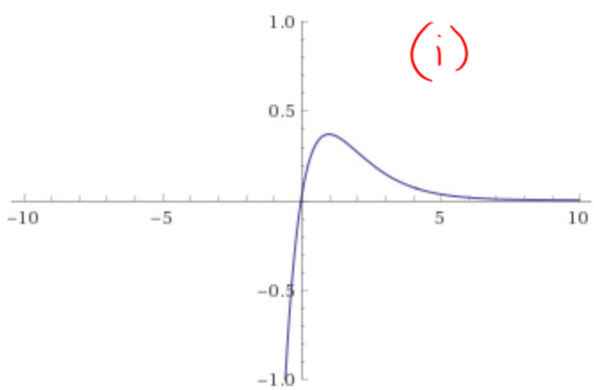
   (iii) $y'' + 6y' + 8y = e^{x}\cos(2x)$ $\qquad$ $\underline{Ae^{x}\cos 2x + Be^{x}\sin 2x}$

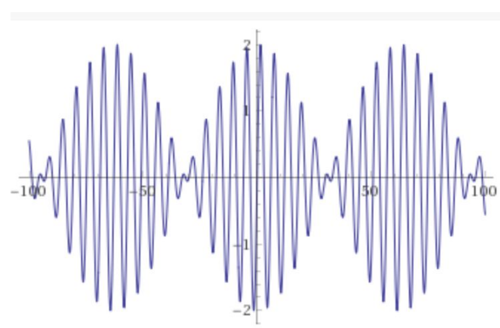
   (b) For each of the solutions y_p above, indicate which plot or plots below are possible graphs of that y_p. No explanation is required. You may pick more than one.



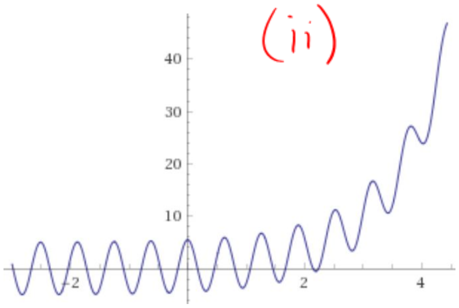


(i)

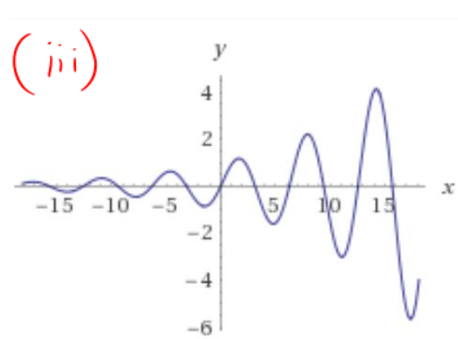




(ii)



(iii)



5. (5 points) A mass of 0.5 kg is attached to a spring with constant 12.5 N/m. There is a damping force such that the damping constant is numerically equal to 5. The mass is released from 0.3 m above equilibrium with a downward velocity of 0.6 m/s.

   (a) Find the position $y(t)$ of the mass as a function of time.

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0$$

$$0.5\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 12.5y = 0$$

aux eqn:

$$0.5n^2 + 5n + 12.5 = 0$$

$$n^2 + 10n + 25 = 0$$

$$(n+5)^2 = 0$$

$$n = -5, -5$$

$$\boxed{y = (c_1 + c_2 t)e^{-5t}}$$

initial conditions:

at $t = 0$, $y = 0.3$

$$0.3 = c_1$$

$\uparrow y$ (positive is up)

at $t = 0$, $\frac{dy}{dt} = -0.6$

$$\frac{dy}{dt} = c_2 e^{-5t} - 5(c_1 + c_2 t)e^{-5t}$$

$$-0.6 = c_2 - 5(0.3)$$

$$c_2 = 0.9$$

$$\boxed{y = (0.3 + 0.9t)e^{-5t}}$$

   (b) This motion is: (choose one)

      (i) overdamped

      (ii) critically damped

      (iii) underdamped