Math 252 – Test 2: Version B

March 6, 2020 **Name:** Solution Set
Instructor: Patricia Wrean

Total: 25 points

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1. (4 points) Find a second solution to the following DE, given that $y_1 = x\cos x$ is a solution.

$$x^2y'' - 2xy' + (x^2+2)y = 0$$

std form: $y'' - \frac{2}{x}y' + \frac{x^2+2}{x^2}y = 0$

reduction of order: $P(x) = -\frac{2}{x}$

$$\int -P(x)\,dx = \int \frac{2}{x}\,dx = 2\ln|x| = \ln x^2$$

$$y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{y_1^2}\,dx$$

$$= x\cos x \int \frac{e^{\ln x^2}}{x^2\cos^2 x}\,dx$$

$$= x\cos x \int \frac{x^2}{x^2\cos^2 x}\,dx$$

$$= x\cos x \int \sec^2 x\,dx$$

$$= x\cos x \tan x$$

$$\boxed{y_2 = x\sin x}$$

2. (6 points) Consider the following DE.

$$y'' + 6\,y' + cy = 0$$

(a) Solve this DE for the following values of c. Give exact answers.

(i) $c = 10$

(ii) $c = 9$

i) $y'' + 6y' + 10y = 0$

$m^2 + 6m + 10 = 0$

$m = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \dfrac{-6 \pm 2i}{2} = -3 \pm i$

$y = e^{-3x}(C_1 \cos x + C_2 \sin x)$

ii) $y'' + 6y' + 9y = 0$

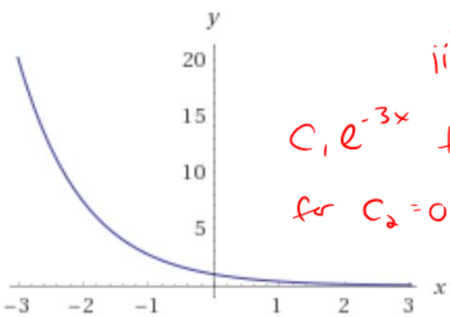
$m^2 + 6m + 9 = 0$

$(m+3)^2 = 0$

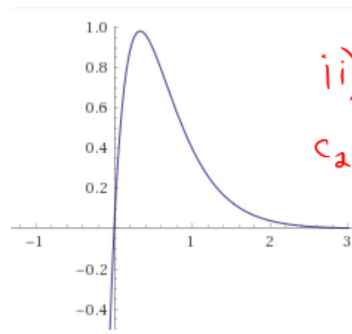
$m = -3, -3$

$y = (C_1 + C_2 x)e^{-3x}$

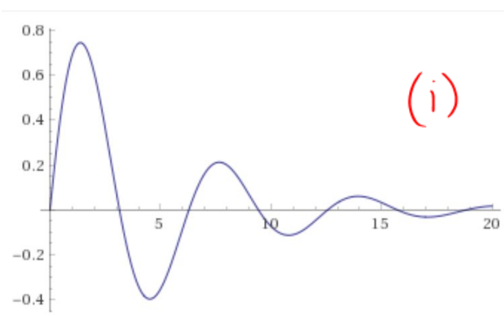
(b) For each of the solutions above, indicate which plot or plots below are possible graphs of that solution. No explanation is required. You may pick more than one.



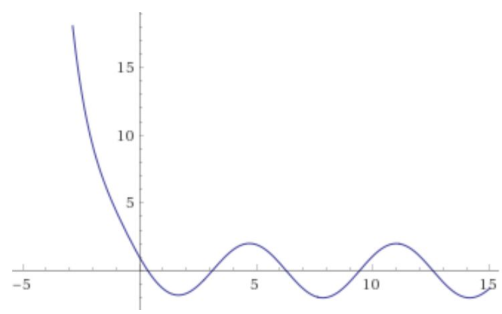
ii) $C_1 e^{-3x}$ term for $C_2 = 0$



ii) $C_2 x e^{-3x}$ for $C_1 = 0$



(i)



3. (6 points) Use the method of variation of parameters to solve the following DE.

$$x^2y'' - 2xy' + 2y = x^5$$

y_c: Cauchy-Euler: aux eqn

$$m(m-1) - 2m + 2 = 0$$
$$m^2 - 3m + 2 = 0$$
$$(m-1)(m-2) = 0$$
$$m = 1, 2$$

$$\boxed{y_c = C_1 x + C_2 x^2}$$

y_p: var of parameters with $y_1 = x$, $y_2 = x^2$

to find $f(x)$, need DE in standard form:

$$y'' - \frac{2}{x} y' + \frac{2}{x^2} y = x^3$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = 2x^2 - x^2 = x^2$$

$$W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix} = \begin{vmatrix} 0 & x^2 \\ x^3 & 2x \end{vmatrix} = -x^5$$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix} = \begin{vmatrix} x & 0 \\ 1 & x^3 \end{vmatrix} = x^4$$

$$u_1 = \int \frac{W_1}{W} dx = \int \frac{-x^5}{x^2} dx = \int -x^3 dx = -\frac{1}{4} x^4$$

$$u_2 = \int \frac{W_2}{W} dx = \int \frac{x^4}{x^2} dx = \int x^2 dx = \frac{1}{3} x^3$$

$$y_p = u_1 y_1 + u_2 y_2$$
$$= -\frac{1}{4} x^4 \cdot x + \frac{1}{3} x^3 \cdot x^2 = \frac{1}{12} x^5$$

$$\boxed{y = y_c + y_p = C_1 x + C_2 x^2 + \frac{1}{12} x^5}$$

4. (3 points) State the form of the particular solution y_p for the following. Leave your answer with undetermined coefficients. Please note that the complementary solution for the homogeneous equation is $y_c = C_1 + C_2 e^{6x}$.

(i) $y'' - 6y' = -2e^{3x}$ Ae^{3x}

(ii) $y'' - 6y' = 5x^2$ ← bad case $Ax^3 + Bx^2 + Cx$

(iii) $y'' - 6y' = -3e^{6x} + \cos 2x$ $Axe^{6x} + B\cos 2x + C\sin 2x$

↑ bad case

5. (6 points) A mass of 0.4 kg is attached to a spring with constant 3.6 N/m. You may assume that there is no damping. The mass is released at 0.5 m above equilibrium with an downward velocity of 0.9 m/s.

(a) Find the position $y(t)$ of the mass as a function of time.

$$m\frac{d^2y}{dt^2} + \cancel{b\frac{dy}{dt}}^{0} + ky = 0$$

$$0.4\frac{d^2y}{dt^2} + 3.6y = 0$$

aux eqn: $0.4m^2 + 3.6 = 0$

$$m^2 + 9 = 0$$

$$m = \pm 3i$$

$$\boxed{y = C_1 \cos 3t + C_2 \sin 3t}$$

$\uparrow y$ (positive is up)

initial conditions:

at $t=0$, $y = 0.5$

$$0.5 = C_1$$

at $t=0$, $\frac{dy}{dt} = -0.9$

$$\frac{dy}{dt} = -3C_1 \sin 3t + 3C_2 \cos 3t$$

$-0.9 = 3C_2$ so $C_2 = -0.3$

$$\boxed{y = 0.5\cos 3t - 0.3\sin 3t}$$

(b) Give the amplitude and period of this motion. You do not need to calculate the phase shift.

$$A = \sqrt{C_1^2 + C_2^2}$$

$$= \sqrt{0.5^2 + (-0.3)^2}$$

$$= 0.640312 = 0.6$$

amplitude: 0.6 m

period: $\frac{2\pi}{3}$ seconds