Term: Winter, ~~2020~~ 2021

Name: Solution Set

Instructor: Patricia Wrean

Math 252-DX01
Test 1

Total = $\overline{20}$

- **Show your work. All of the work on this test must be your own.** While writing this test, you may not consult any other person, website, or other resource not listed below. If you have a question during the test, you may email me.
- Here is a list of the resources that you are allowed to use during this test:
  - your own notes
  - lecture notes, videos, handouts, practice questions, and practice tests from either my website at http://wrean.ca/math252 or the Math 252 websites of any of the other instructors linked on the landing page of my site
  - your textbook (Zill), or any of the texts listed on the Textbook page at http://wrean.ca/math252/math252_textbook.htm
  - the Math 252 D2L website
  - the Math 252 WeBWorK online homework site
  - a scientific calculator. You may not use a calculator with graphing capability. If you like, you may use a scientific calculator app like the one at Desmos: https://www.desmos.com/scientific
  - a handy reference is the Math 252 Formula Sheet at http://wrean.ca/math252/tests/math252_formula.pdf
  - if you have questions during the test, you may email me
- To submit this test, please use the Dropbox feature in the Assignments tab of D2L. Please assemble your answers into a single PDF or Word document, unless you've made other arrangements with me beforehand. Helpful software:
  - Genius Scan app at https://www.thegrizzlylabs.com/genius-scan/
  - CombinePDF at https://combinepdf.com/

GOOD LUCK!

1. (6 points) Consider the following DE.

$$2y' + y = 2e^{-x}, \qquad y(0) = 5$$

(a) Solve this DE, giving an explicit solution.

(b) Is your answer to part (a) a particular solution? Circle one: Yes / No

because all constants have known values

1

$y' + \frac{1}{2}y = e^{-x}$ 1 linear 1st order in standard form

$P(x) = \frac{1}{2}$

$IF = e^{\int P(x)\,dx} = e^{\int \frac{1}{2}dx} = e^{\frac{1}{2}x}$ 1

$y' e^{\frac{1}{2}x} + \frac{1}{2} y e^{\frac{1}{2}x} = e^{-\frac{1}{2}x}$

$\frac{d}{dx}\left(y e^{\frac{1}{2}x}\right) = e^{-\frac{1}{2}x}$

$\int \frac{d}{dx}\left(y e^{\frac{1}{2}x}\right)dx = \int e^{-\frac{1}{2}x}\,dx$

$y e^{\frac{1}{2}x} = -2e^{-\frac{1}{2}x} + C$ 1

but $y(0) = 5$

$5e^0 = -2e^0 + c$

$c = 7$ 1

$y e^{\frac{1}{2}x} = -2e^{-\frac{1}{2}x} + 7$

$\boxed{y = -2e^{-x} + 7e^{-\frac{1}{2}x}}$ 1

2. (4 points) Solve the following DE.

$$(y^3 + xy^2)dx - x^2ydy = 0$$

homogeneous of degree 3 (or Bernoulli - see next page)

dy is simpler so let $\begin{cases} y = ux \\ dy = udx + xdu \end{cases}$ 1

then $(y^3 + xy^2)dx - x^2ydy = 0$

$$(u^3x^3 + xu^2x^2)\,dx - x^2ux(udx + xdu) = 0 \quad 1$$

$$u^3x^3dx + \cancel{u^2x^3dx} - \cancel{u^2x^3}dx - ux^4du = 0$$

$$u^3x^3dx = ux^4du \quad \text{separable} \quad 1$$

$$\frac{dx}{x} = \frac{du}{u^2}$$

$$\ln|x| = -u^{-1} + C$$

$$\boxed{\ln|x| = -\frac{x}{y} + C} \quad 1$$

if you insist on solving explicitly

$$\ln|x| - C = -\frac{x}{y}$$

$$\boxed{y = \frac{-x}{\ln|x| - C}} \quad \text{or} \quad \boxed{y = \frac{-x}{\ln|x| + C_1}}$$

2 continued | Bernoulli solution

$$(y^3 + xy^2)\,dx - x^2 y\,dy = 0$$

$$x^2 y\,dy = (y^3 + xy^2)\,dx$$

$$\frac{dy}{dx} = \frac{y^2}{x^2} + \frac{y}{x}$$

$$\frac{dy}{dx} - \frac{y}{x} = \frac{1}{x^2} y^2$$

Bernoulli with $n=2$

so let

$$v = y^{1-n} = y^{-1} \quad 1$$

$$\frac{dv}{dx} = -\frac{1}{y^2}\frac{dy}{dx}$$

mult DE by this

$$-\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{x}\frac{1}{y} = -\frac{1}{x^2}$$

$$1 \quad \frac{dv}{dx} + \frac{1}{x} v = -\frac{1}{x^2}$$

1st order linear

$$IF = e^{\int P(x)\,dx} = e^{\int \frac{1}{x}dx} = e^{\ln|x|} = |x| = x \quad \text{(can drop || in IF)}$$

1

$$x\frac{dv}{dx} + v = -\frac{1}{x}$$

$$\frac{d}{dx}(vx) = -\frac{1}{x}$$

$$\int \frac{d}{dx}(vx)\,dx = \int -\frac{1}{x}\,dx$$

$$vx = -\ln|x| + C$$

$$\boxed{\frac{x}{y} = -\ln|x| + C} \quad 1$$

as before

3. (5 points) Consider the following DE.

$$1 + y^2 + k(x+1)y\frac{dy}{dx} = 0$$

(a) Find the value of k for which this DE is exact.

$$\underbrace{(1+y^2)}_{M}dx + \underbrace{k(x+1)y}_{N}\,dy = 0$$

$$\frac{\partial M}{\partial y} = 2y \qquad \frac{\partial N}{\partial x} = ky \qquad \text{so } \boxed{k=2}$$

1

1

(b) Solve this DE using the value of k you found in part (a).

$$(1+y^2)\,dx + 2(x+1)y\,dy = 0$$

$$f = \int (1+y^2)\,dx \quad \underline{\text{and}} \quad f = \int 2(x+1)y\,dy$$

$$f = x + xy^2 + g(y) \qquad f = xy^2 + y^2 + h(x)$$

1

1

$$f = xy^2 + x + y^2 = C$$

$$\boxed{xy^2 + x + y^2 = C}$$

1

4. (5 points) Water is pumped continuously from an underground tank at a rate proportional to the amount of water left in the tank. Initially there were two thousand litres in the tank; exactly six days later only half of that amount remains. You want to shut off the pump when there is only one hundred litres left so that you don't damage the pump. When do you need to shut off the pump?

Start by setting up and solving the associated DE. Show your work.

let $V =$ amount of water left in the tank

1 $\frac{dV}{dt} = kV$ ← separable and linear first-order

$$\frac{dV}{dt} - kV = 0$$

$$IF = e^{\int P(t)\,dt} = e^{\int -k\,dt} = e^{-kt}$$

$$\frac{dV}{dt}e^{-kt} - kVe^{-kt} = 0$$

$$\frac{d}{dt}\left(Ve^{-kt}\right) = 0$$

$$Ve^{-kt} = C$$

$$V = Ce^{kt}$$

if DE was $\frac{dV}{dt} = -kV$ then $V = Ce^{-kt}$ here

but at $t = 0$, $V = 2000$ L

$$V = 2000e^{kt}$$ 1

after $t = 6$ days, $V = 1000$ L

$$1000 = 2000e^{k \cdot 6}$$

$$\frac{1}{2} = e^{6k}$$

$$\ln(1/2) = 6k$$

$$k = \frac{1}{6}\ln(1/2)$$ 1

if DE was $\frac{dV}{dt} = -kV$, then get $k = -\frac{1}{6}\ln(1/2)$ here time to reach 100 L:

$$V = 2000 e^{kt}$$

$$100 = 2000 e^{kt}$$

1

$$\frac{1}{20} = e^{kt}$$

$$\ln\left(\frac{1}{20}\right) = kt$$

$$t = \frac{1}{k} \ln\left(\frac{1}{20}\right) = \frac{6 \ln\left(\frac{1}{20}\right)}{\ln\left(\frac{1}{2}\right)}$$

$$\approx 25.9316$$

$$\boxed{t \approx 26 \text{ days}}$$

1