Term: Winter, 2021

Instructor: Patricia Wrean

Name: Solution Set

Math 252-DX01
Test 2

Total = $\frac{\quad}{20}$

- **Show your work. All of the work on this test must be your own.** While writing this test, you may not consult any other person, website, or other resource not listed below. If you have a question during the test, you may email me.
- Here is a list of the resources that you are allowed to use during this test:
  - your own notes
  - lecture notes, videos, handouts, practice questions, and practice tests from either my website at http://wrean.ca/math252 or the Math 252 websites of any of the other instructors linked on the landing page of my site
  - your textbook (Zill), or any of the texts listed on the Textbook page at http://wrean.ca/math252/math252_textbook.htm
  - the Math 252 D2L website
  - the Math 252 WeBWorK online homework site
  - a scientific calculator. You may not use a calculator with graphing capability. If you like, you may use a scientific calculator app like the one at Desmos: https://www.desmos.com/scientific
  - a handy reference is the Math 252 Formula Sheet at http://wrean.ca/math252/tests/math252_formula.pdf
  - if you have questions during the test, you may email me
- To submit this test, please use the Dropbox feature in the Assignments tab of D2L. Please assemble your answers into a single PDF or Word document, unless you've made other arrangements with me beforehand. Helpful software:
  - Genius Scan app at https://www.thegrizzlylabs.com/genius-scan/
  - CombinePDF at https://combinepdf.com/

GOOD LUCK!

1. (4 points) Find a second solution to the following DE, given that $y_1 = x$ is a solution. You may assume that $x > 0$.

$$x^2y'' - x(x+2)y' + (x+2)y = 0$$

standard form:

$$y'' - \frac{x(x+2)}{x^2}y' + \frac{(x+2)}{x^2}y = 0$$

$$P(x) = -\frac{x(x+2)}{x^2} = -\left(1 + \frac{2}{x}\right) \quad 1$$

$$e^{-\int P(x)dx} = e^{\int (1+2/x)dx}$$

$$= e^{x + 2\ln x} \quad 1$$

– can drop abs value because $x > 0$

$$= e^x e^{\ln x^2}$$

$$= x^2 \ln x \quad 1$$

$$y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{x^2}dx$$

$$= x \int \frac{x^2 e^x}{x^2}dx$$

$$= x \int e^x dx$$

$$= xe^x \quad 1$$

and if you like,

$$y = c_1 y_1 + c_2 y_2$$

$$\boxed{y = c_1 x + c_2 xe^x}$$

can omit

2. (6 points) Solve the following DE.

$$y'' + 4y' + 3y = 5e^{-x} + 7$$

y_c: aux eqn $m^2 + 4m + 3 = 0$

$(m+3)(m+1) = 0$

$m = -3, -1$

$\boxed{y_c = C_1 e^{-3x} + C_2 e^{-x}}$ 1

y_p: RHS is $g(x) = 5e^{-x} + 7$

let $y_p = A\cancel{e^{-x}} + B$ bad case 1

$y_p = Axe^{-x} + B$

if no B, −1

1 so $y_p' = Ae^{-x} - Axe^{-x}$

$y_p'' = -Ae^{-x} - Ae^{-x} + Axe^{-x} = -2Ae^{-x} + Axe^{-x}$

sub back into DE:

$$y'' + 4y' + 3y = 5e^{-x} + 7$$

$$-2Ae^{-x} + \cancel{Axe^{-x}} + 4(Ae^{-x} - \cancel{Axe^{-x}}) + 3(\cancel{Axe^{-x}} + B) = 5e^{-x} + 7$$

$$2Ae^{-x} + 3B = 5e^{-x} + 7$$

$2A = 5$ $\quad$ $3B = 7$ 1

$A = 5/2$ $\quad$ $B = 7/3$

$\boxed{y_p = Axe^{-x} + B = \frac{5}{2}xe^{-x} + \frac{7}{3}}$ 1

$\boxed{y = y_p + y_c = C_1 e^{-3x} + C_2 e^{-x} + \frac{5}{2}xe^{-x} + \frac{7}{3}}$ 1

3. (6 points) Use variation of parameters to solve the following DE for $x > 0$.

$$x^2y'' + x\,y' - 4y = \sqrt{x}$$

y_c: Cauchy Euler

aux eqn = $m(m-1) + m - 4 = 0$

$m^2 - 4 = 0$

$m = \pm 2$

$$\boxed{y_c = c_1 x^2 + c_2 x^{-2}}$$

1

y_p: variation of parameters

let $y_1 = x^2$

$y_2 = x^{-2}$

std form: $y'' + \frac{1}{x} y' - \frac{4}{x^2} y = \frac{\sqrt{x}}{x^2}$

so $\boxed{f(x) = x^{-3/2}}$

1

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x^2 & x^{-2} \\ 2x^2 & -2x^{-3} \end{vmatrix} = -2x^{-1} - 2x^{-1} = -4x^{-1}$$

$$W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix} = \begin{vmatrix} 0 & x^{-2} \\ x^{-3/2} & -2x^{-3} \end{vmatrix} = -x^{-7/2}$$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix} = \begin{vmatrix} x^2 & 0 \\ 2x^2 & x^{-3/2} \end{vmatrix} = x^{1/2}$$

1

$$u_1 = \int \frac{W_1}{W}\,dx = \int \frac{-x^{-7/2}}{-4x^{-1}}\,dx = \int \frac{1}{4} x^{-5/2}\,dx = -\frac{2}{3}\cdot\frac{1}{4} x^{-3/2} = -\frac{1}{6} x^{-3/2}$$

$$u_2 = \int \frac{W_2}{W}\,dx = \int \frac{x^{1/2}}{-4x^{-1}}\,dx = \int -\frac{1}{4} x^{3/2}\,dx = \frac{2}{5}\left(-\frac{1}{4}\right) x^{5/2} = -\frac{1}{10} x^{5/2}$$

1

$$y_p = u_1 y_1 + u_2 y_2 = -\tfrac{1}{6} x^{-3/2} x^2 - \frac{1}{10} x^{5/2} x^{-2}$$

$$= -\tfrac{1}{6} x^{1/2} - \tfrac{1}{10} x^{1/2} = -\frac{4}{15} x^{1/2}$$

1

$$\boxed{y = y_c + y_p = c_1 x^2 + c_2 x^{-2} - \frac{4}{15}\sqrt{x}}$$

1

4. (4 points) When a mass of 0.25 kg is attached to an ideal spring, the spring stretches by a length of 1.25 m. The mass-spring system is allowed to come to equilibium, and then set in motion with some initial conditions. The air resistance is proportional to the speed of the mass with damping constant equal to b.

Use 9.8 m/s^2 for the acceleration due to gravity.

(a) Give the differential equation that shows the relationship between the position $y(t)$ of the mass and the time elapsed t.

at equilibrium

F_{spring}

F_g

$F_g = F_{spring}$

$mg = kx$

$(0.25)(9.8) = k(1.25)$

$k = 1.96$

1

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = 0$$

$$\boxed{0.25\frac{d^2y}{dt^2} + b\frac{dy}{dt} + 1.96y = 0}$$

1

(b) For what numerical values of the damping constant b will $y(t)$ have the shape shown in the graph below? (Don't bother with the units.)

aux eqn:

$0.25n^2 + bn + 1.96 = 0$

$$n = \frac{-b \pm \sqrt{b^2 - 4(0.25)(1.96)}}{0.5}$$

to get complex n, need

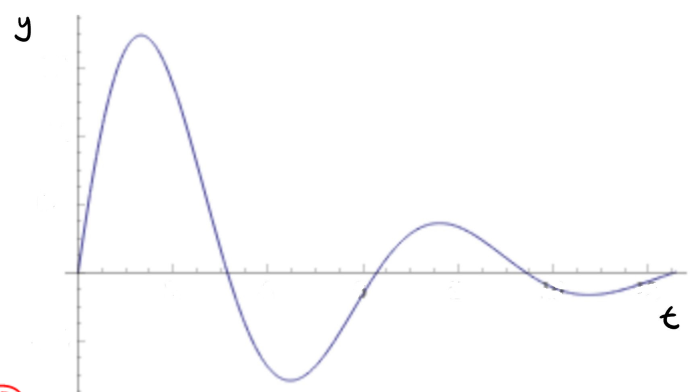
$b^2 - 4(0.25)(1.96) < 0$

$b^2 < 1.96$

$b < 1.4$

1

$$\boxed{0 < b < 1.4}$$



note: b cannot be zero or negative then you wouldn't get decaying amplitude

$-\frac{1}{2}$

(c) The motion described by the graph in part (b) is: (choose one)

(i) overdamped

(ii) critically damped

(iii) underdamped

1