Term: Winter, 2021

Name: Solution Set

Instructor: Patricia Wrean

Math 252-DX01
Test 3

Total = $\frac{\quad}{20}$

- **Show your work. All of the work on this test must be your own.** While writing this test, you may not consult any other person, website, or other resource not listed below. If you have a question during the test, you may email me.
- Here is a list of the resources that you are allowed to use during this test:
  - your own notes
  - lecture notes, videos, handouts, practice questions, and practice tests from either my website at http://wrean.ca/math252 or the Math 252 websites of any of the other instructors linked on the landing page of my site
  - your textbook (Zill), or any of the texts listed on the Textbook page at http://wrean.ca/math252/math252_textbook.htm
  - the Math 252 D2L website
  - the Math 252 WeBWorK online homework site
  - a scientific calculator. You may not use a calculator with graphing capability. If you like, you may use a scientific calculator app like the one at Desmos: https://www.desmos.com/scientific
  - a handy reference is the Math 252 Formula Sheet at http://wrean.ca/math252/tests/math252_formula.pdf
  - if you have questions during the test, you may email me
- To submit this test, please use the Dropbox feature in the Assignments tab of D2L. Please assemble your answers into a single PDF or Word document, unless you've made other arrangements with me beforehand. Helpful software:
  - Genius Scan app at https://www.thegrizzlylabs.com/genius-scan/
  - CombinePDF at https://combinepdf.com/

GOOD LUCK!

1. (6 points) Consider the power series solution for the following differential equation about the ordinary point $x = 0$.

$$(1 - x^2)y'' + y = 0$$

(a) For what values of x can we guarantee that the series converge?

(b) Find the recurrence relation for the coefficients of the series. Do not bother to calculate any coefficients.

a) $y'' + \frac{1}{1-x^2} y = 0$ so $x \neq \pm 1$ $\boxed{-1 < x < 1}$

1

$\boxed{-1 \leq x \leq 1 \quad -\tfrac{1}{2}}$

b) let $y = \sum_{n=0}^{\infty} c_n x^n$

$$y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$$

sub into DE: $(1-x^2) y'' + y = 0$

$$(1-x^2) \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=0}^{\infty} c_k x^k = 0$$

sub into DE, multiply through 1

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} \underbrace{- \sum_{n=2}^{\infty} n(n-1) c_n x^n + \sum_{n=0}^{\infty} c_k x^k}_{\text{let } k=n} = 0$$

let $k = n-2$
$k+2 = n$
when $n=2$, $k=0$

1 reindex

$$\sum_{k=0}^{\infty} (k+2)(k+1) c_{k+2} x^k - \sum_{k=2}^{\infty} k(k-1) c_k x^k + \sum_{k=0}^{\infty} c_k x^k = 0$$

write out extra terms 1

$$\underset{k=0}{2c_2} + \underset{k=1}{6c_3 x} + \sum_{k=2}^{\infty} (k+2)(k+1) c_{k+2} x^k - \sum_{k=2}^{\infty} k(k-1) c_k x^k + \underset{k=0}{c_0} + \underset{k=1}{c_1 x} + \sum_{k=2}^{\infty} c_k x^k = 0$$

$$(2c_2 + c_0) + (6c_3 + c_1)x + \sum_{k=2}^{\infty} \left[(k+2)(k+1) c_{k+2} - k(k-1) c_k + c_k \right] x^k = 0$$

so $2c_2 + c_0 = 0$
$6c_3 + c_1 = 0$
$(k+2)(k+1) c_{k+2} - k(k-1) c_k + c_k = 0$

1

therefore $c_2 = -\frac{1}{2} c_0$
$c_3 = -\frac{1}{6} c_1$

1

no k = $-\tfrac{1}{2}$ $\rightarrow$

$$\boxed{c_{k+2} = \frac{k(k-1) - 1}{(k+2)(k+1)} c_k \text{ for } k = 2, 3, 4, \ldots}$$

recurrence relation

2. (4 points) The power series solution for a particular differential equation has the following relationships between the coefficients.

$$\begin{cases} C_2 = \dfrac{-3C_0}{2} \\ \\ C_{k+2} = \dfrac{-(k+3)C_k}{(k+2)(k+1)} \quad \text{for } k = 1, 2, 3, \ldots \end{cases}$$

From this starting point, find two power series solutions of that differential equation about the ordinary point $x = 0$. Give the first three non-zero terms of each series (if they exist) and the general solution.

let $C_0 = 1$ and $C_1 = 0$ to get y_1

then $C_2 = -\frac{3}{2} C_0 = -3/2$

when $k=1$, $C_3 = -\frac{4}{3 \cdot 2} C_1 = 0$

when $k=2$, $C_4 = \frac{-5}{4 \cdot 3} C_2 = \frac{-5}{4 \cdot 3}\left(-3/2\right) = +\frac{5}{8}$

let $C_0 = 0$ and $C_1 = 1$ to get y_2

then $C_2 = -3/2\, C_0 = 0$

$\boxed{k=1}$ $C_3 = -\frac{4}{3 \cdot 2} C_1 = -\frac{2}{3}$

$\boxed{k=2}$ $C_4 = -\frac{5}{12} C_2 = 0$

$\boxed{k=3}$ $C_5 = \frac{-6}{5 \cdot 4} C_3 = \frac{-3}{5 \cdot 2}\left(-2/3\right) = +\frac{1}{5}$

then $y = C_0 y_1 + C_1 y_2$

where $y_1 = 1 - 3/2\, x^2 + 5/8\, x^4 + \ldots$

and $y_2 = x - \frac{2}{3} x^3 + \frac{1}{5} x^5 + \ldots$

1 write C_3 in terms of C_1, C_4 in terms of C_2

1 sub C_0, C_1, and C_2 into C_2, C_3, C_4 respectively

1 correct approach to y_1 / y_2: either let $C_0 = 1$ and $C_1 = 0$, etc, or group C_0 terms into y_1

1 final answer

if no $y =$, only y_1 and y_2 $-1/2$

3. (5 points) Find the following Laplace and inverse Laplace transforms.

(a) Find the inverse Laplace transform $\mathscr{L}^{-1}\{F(s)\}$ for

$$F(s) = \frac{3s+4}{s^2+4}$$

$$F(s) = \frac{3s}{s^2+4} + 2\,\frac{2}{s^2+4} \qquad 1$$

$$\boxed{f(t) = 3\cos 2t + 2\sin 2t} \qquad 1$$

incorrect coeffs $-\frac{1}{2}$

(b) Find the Laplace transform $\mathscr{L}\{f(t)\}$ for

$$f(t) = \begin{cases} t & \text{if } 0 \le t < 6 \\ 0 & \text{if } t \ge 6 \end{cases}$$

$$f(t) = t\left[1 - \mathcal{U}(t-6)\right] \qquad 1$$

now $\mathscr{L}\{f(t)\,\mathcal{U}(t-a)\} = e^{-as}\mathscr{L}\{f(t+a)\}$

so $\mathscr{L}\{f(t)\} = \mathscr{L}\{t\} - \mathscr{L}\{t\,\mathcal{U}(t-6)\}$

$$= \frac{1}{s^2} - e^{-6s}\mathscr{L}\{t+6\}$$

$$\boxed{F(s) = \underbrace{\frac{1}{s^2}}_{1} - \underbrace{e^{-6s}\left(\frac{1}{s^2} + \frac{6}{s}\right)}_{1}}$$

4. (5 points) Use the Laplace transform to solve the following initial-value problem.

$$y'' - 6y' + 8y = e^{-3t}, \qquad y(0) = 0, y'(0) = 0$$

$$[s^2 Y(s) - s\cancel{y(0)}^{0} - \cancel{y'(0)}^{0}] - 6[sY(s) - \cancel{y(0)}^{0}] + 8Y(s) = \frac{1}{s+3} \quad 1$$

$$(s^2 - 6s + 8)\, Y(s) = \frac{1}{s+3}$$

$$(s-2)(s-4)\, Y(s) = \frac{1}{s+3}$$

$$Y(s) = \frac{1}{(s-2)(s+3)(s-4)} \quad 1$$

partial fractions:

$$\frac{1}{(s-2)(s+3)(s-4)} = \frac{A}{s-2} + \frac{B}{s+3} + \frac{C}{s-4}$$

$$A = \left.\frac{1}{(s+3)(s-4)}\right|_{s=2} = \frac{1}{5(-2)} = -\frac{1}{10}$$

$$B = \left.\frac{1}{(s-2)(s-4)}\right|_{s=-3} = \frac{1}{(-5)(-7)} = \frac{1}{35}$$

$$C = \left.\frac{1}{(s-2)(s+3)}\right|_{s=4} = \frac{1}{2 \cdot 7} = \frac{1}{14}$$

$$Y(s) = -\frac{1}{10(s-2)} + \frac{1}{35(s+3)} + \frac{1}{14(s-4)}$$

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$$\boxed{y(t) = -\frac{1}{10}e^{2t} + \frac{1}{35}e^{-3t} + \frac{1}{14}e^{4t}} \quad 1$$