Term: Winter, 2021

Name: Solution Set

Instructor: Patricia Wrean

Math 252-DX02
Test 3

Total = $\overline{20}$

- **Show your work. All of the work on this test must be your own.** While writing this test, you may not consult any other person, website, or other resource not listed below. If you have a question during the test, you may email me.
- Here is a list of the resources that you are allowed to use during this test:
  - your own notes
  - lecture notes, videos, handouts, practice questions, and practice tests from either my website at http://wrean.ca/math252 or the Math 252 websites of any of the other instructors linked on the landing page of my site
  - your textbook (Zill), or any of the texts listed on the Textbook page at http://wrean.ca/math252/math252_textbook.htm
  - the Math 252 D2L website
  - the Math 252 WeBWorK online homework site
  - a scientific calculator. You may not use a calculator with graphing capability. If you like, you may use a scientific calculator app like the one at Desmos: https://www.desmos.com/scientific
  - a handy reference is the Math 252 Formula Sheet at http://wrean.ca/math252/tests/math252_formula.pdf
  - if you have questions during the test, you may email me
- To submit this test, please use the Dropbox feature in the Assignments tab of D2L. Please assemble your answers into a single PDF or Word document, unless you've made other arrangements with me beforehand. Helpful software:
  - Genius Scan app at https://www.thegrizzlylabs.com/genius-scan/
  - CombinePDF at https://combinepdf.com/

GOOD LUCK!

1. (6 points) Consider the power series solution for the following differential equation about the ordinary point $x = 0$.

$$(1-x)y'' + y = 0$$

(a) For what values of x can we guarantee that the series converge?

(b) Find the recurrence relation for the coefficients of the series. Do not bother to calculate any coefficients.

a) $y'' + \frac{1}{1-x} y = 0$ so $x \neq 1$ $\boxed{-1 < x < 1}$

(interval must be symmetric about $x=0$)

1

$-1 \leq x \leq 1$ $-\frac{1}{2}$

b) let $y = \sum_{n=0}^{\infty} c_n x^n$

$y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$

$y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$

sub into DE: $(1-x)y'' + y = 0$

$$(1-x)\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} c_k x^k = 0$$

sub into DE, multiply through 1

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)c_n x^{n-1} + \sum_{n=0}^{\infty} c_k x^k = 0$$

let $k = n-2$, $k+2 = n$, when $n=2$, $k=0$

let $k = n-1$, $k+1 = n$, when $n=2$, $k=1$

let $x = n$

$$\sum_{k=0}^{\infty} (k+2)(k+1)c_{k+2}x^k - \sum_{k=1}^{\infty}(k+1)k\,c_{k+1}x^k + \sum_{k=0}^{\infty} c_k x^k = 0$$

1 reindex

write out extra terms 1

$$\underset{k=0}{2c_2} + \sum_{k=1}^{\infty}(k+2)(k+1)c_{k+2}x^k - \sum_{k=1}^{\infty} k(k+1)c_{k+1}x^k + \underset{k=0}{c_0} + \sum_{k=1}^{\infty} c_k x^k = 0$$

$$(2c_2 + c_0) + \sum_{k=1}^{\infty}\left[(k+2)(k+1)c_{k+2} - k(k+1)c_{k+1} + c_k\right]x^k = 0$$

so $2c_2 + c_0 = 0$

$(k+2)(k+1)c_{k+2} - k(k+1)c_{k+1} + c_k = 0$ 1

therefore $c_2 = -\frac{1}{2}c_0$

no $k=$ $-\frac{1}{2}$ $\longrightarrow$ $\boxed{c_{k+2} = \frac{k(k+1)c_{k+1} - c_k}{(k+2)(k+1)} \quad \text{for } k = 1, 2, 3, \ldots}$ 1

recurrence relation

2. (4 points) The power series solution for a particular differential equation has the following relationships between the coefficients.

$$\begin{cases} C_2 = \dfrac{2C_0 - 3C_1}{2} \\[2ex] C_{k+2} = \dfrac{2C_k - (k+1)(k+3)C_{k+1}}{(k+2)(k+1)} & \text{for } k = 1, 2, 3, \ldots \end{cases}$$

From this starting point, find two power series solutions of that differential equation about the ordinary point $x = 0$. Give the first three non-zero terms of each series (if they exist) and the general solution.

let $C_0 = 1$ and $C_1 = 0$

to get y_1

$$C_2 = \frac{2C_0 - 3C_1}{2} = \frac{2(1) - 3(0)}{2} = 1$$

$k=1$ $$C_3 = \frac{2C_1 - 2\cdot 4C_2}{3\cdot 2} = \frac{0-8}{6} = -\frac{4}{3}$$

let $C_0 = 0$ and $C_1 = 1$ to get y_2

$$C_2 = \frac{2C_0 - 3C_1}{2} = \frac{0-3}{2} = -\frac{3}{2}$$

$$C_3 = \frac{2C_1 - 8C_2}{6} = \frac{2(1) - 8(-3/2)}{6}$$

$$= \frac{14}{6} = 7/3$$

then $y = C_0 y_1 + C_1 y_2$

where $y_1 = 1 + x^2 - 4/3\, x^3 + \ldots$

$y_2 = x - 3/2\, x^2 + 7/3\, x^3 + \ldots$

1 write C_3 in terms of C_1, C_4 in terms of C_2	1 sub C_0, C_1, and C_2 into C_2, C_3, C_4 respectively	1 correct approach to y_1 / y_2: either let $C_0 = 1$ and $C_1 = 0$, etc, or group C_0 terms into y_1	1 find answer; if no $y=$, only y_1 and y_2 $-1/2$

3. (5 points) Find the following Laplace and inverse Laplace transforms.

(a) Find the Laplace transform $\mathscr{L}\{f(t)\}$ for

$$f(t) = (t-1)^2$$

$$= t^2 - 2t + 1$$

1

$$\boxed{\mathcal{L}\{f(t)\} = \frac{2!}{s^3} - \frac{2}{s^2} + \frac{1}{s}}$$

1

(b) Find the inverse Laplace transform $\mathscr{L}^{-1}\{F(s)\}$ for

$$F(s) = \frac{(1+e^{-s})^2}{s+3}$$

$$F(s) = \frac{1 + 2e^{-s} + e^{-2s}}{s+3}$$

1

$$= \frac{1}{s+3} + \frac{2e^{-s}}{s+3} + \frac{e^{-2s}}{s+3}$$

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathcal{U}(t-a)$$

$$\boxed{f(t) = e^{-3t} + \underbrace{2e^{-3(t-1)}\mathcal{U}(t-1) + e^{-3(t-2)}\mathcal{U}(t-2)}_{1}}$$

1

4. (5 points) Use the Laplace transform to solve the following initial-value problem.

$$y'' - 6y' + 9y = te^{3t}, \qquad y(0) = 2, y'(0) = 12$$

$$\left[s^2 Y(s) - s\,y(0) - y'(0)\right] - 6\left[sY(s) - y(0)\right] + 9Y(s) = \frac{1}{(s-3)^2} \qquad 1$$

$$s^2Y(s) - 6sY(s) + 9Y(s) - 2s - 12 + 12 = \frac{1}{(s-3)^2}$$

$$(s^2 - 6s + 9)\,Y(s) \quad -2s = \frac{1}{(s-3)^2}$$

$$(s-3)^2\,Y(s) = 2s + \frac{1}{(s-3)^2}$$

$$Y(s) = \frac{2s}{(s-3)^2} + \frac{1}{(s-3)^4} \qquad 1$$

partial fractions:

$$\frac{2s}{(s-3)^2} = \frac{A}{s-3} + \frac{B}{(s-3)^2}$$

$$2s = A(s-3) + B$$

$$2s = As - 3A + B$$

so $A = 2$

$-3A + B = 0$

$B = 6$

2

$$Y(s) = \frac{2}{s-3} + \frac{4}{(s-3)^2} + \frac{1}{(s-3)^4}$$

$$y(t) = 2e^{3t} + 4te^{3t} + \frac{t^3}{3!}e^{3t}$$

$$\boxed{y(t) = 2e^{3t} + 6te^{3t} + \frac{1}{6}t^3e^{3t}} \qquad 1$$