Term: Winter, 2021

Name: Solution Set

Instructor: Patricia Wrean

Math 252-DX02
Test 2

Total = $\overline{\quad\ 20}$

- **Show your work. All of the work on this test must be your own.** While writing this test, you may not consult any other person, website, or other resource not listed below. If you have a question during the test, you may email me.
- Here is a list of the resources that you are allowed to use during this test:
  - your own notes
  - lecture notes, videos, handouts, practice questions, and practice tests from either my website at http://wrean.ca/math252 or the Math 252 websites of any of the other instructors linked on the landing page of my site
  - your textbook (Zill), or any of the texts listed on the Textbook page at http://wrean.ca/math252/math252_textbook.htm
  - the Math 252 D2L website
  - the Math 252 WeBWorK online homework site
  - a scientific calculator. You may not use a calculator with graphing capability. If you like, you may use a scientific calculator app like the one at Desmos: https://www.desmos.com/scientific
  - a handy reference is the Math 252 Formula Sheet at http://wrean.ca/math252/tests/math252_formula.pdf
  - if you have questions during the test, you may email me
- To submit this test, please use the Dropbox feature in the Assignments tab of D2L. Please assemble your answers into a single PDF or Word document, unless you've made other arrangements with me beforehand. Helpful software:
  - Genius Scan app at https://www.thegrizzlylabs.com/genius-scan/
  - CombinePDF at https://combinepdf.com/

GOOD LUCK!

1. (5 points) Solve the following DE, given that $y_1 = 1$ is a solution. You may assume that $0 < x < 1$.

$$(1-x^2)y'' + 2xy' = 0$$

Standard form: $y'' + \underbrace{\frac{2x}{1-x^2}}_{P(x)} y' = 0$ 1

so $e^{-\int P(x)dx} = e^{-\int \frac{2x}{1-x^2}dx}$

let $u = 1-x^2$
$du = -2x\,dx$
then integral is
$e^{\int \frac{1}{u}du} = e^{\ln|u|}$

$= e^{\ln|1-x^2|}$ 1

$= e^{\ln(1-x^2)}$ because $0<x<1$

$= (1-x^2)$ 1

$$y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{y_1^2}dx$$

$$= 1 \int \frac{(1-x^2)}{1^2}dx = \int (1-x^2)dx = x - \frac{x^3}{3}$$ 1

and

$$y = c_1 y_1 + c_2 y_2$$

$$\boxed{y = c_1 + c_2\left(x - \frac{x^3}{3}\right)}$$ 1

2. (5 points) Solve the following DE.

$$y'' + y = \sec x \tan x$$

y_c: aux eqn: $m^2 + 1 = 0$

$m = \pm i$

$$\boxed{y_c = C_1 \cos x + C_2 \sin x}$$

1

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y_p: let $y_1 = \cos x$ and $f(x) = \sec x \tan x$

$y_2 = \sin x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = \boxed{1}$$

1

$$W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix} = \begin{vmatrix} 0 & \sin x \\ \sec x \tan x & \cos x \end{vmatrix} = -\sin x \sec x \tan x$$

$$= \frac{-\sin x}{\cos x} \tan x$$

$$= \boxed{-\tan^2 x}$$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix} = \begin{vmatrix} \cos x & 0 \\ -\sin x & \sec x \tan x \end{vmatrix} = \cos x \sec x \tan x$$

$$\boxed{= \tan x}$$

then $u_1 = \int \frac{W_1}{W} dx$

$$= \int -\tan^2 x \, dx = \int (1 - \sec^2 x) dx = x - \tan x$$

1

note: $1 + \tan^2 x = \sec^2 x$

$1 - \sec^2 x = -\tan^2 x$

$u_2 = \int \frac{W_2}{W} dx$

$= \int \tan x \, dx = \ln|\sec x|$ (or $-\ln|\cos x|$ if you prefer) then

$$y_p = u_1 y_1 + u_2 y_2$$

1

$$= (x - \tan x)\cos x + \ln|\sec x| \cdot \sin x$$

$$= x\cos x - \sin x + \sin x \ln|\sec x|$$

and $y = y_c + y_p$

collect like terms and let $C_3 = C_2 - 1$

1

$$= C_1 \cos x + C_2 \sin x + x\cos x - \sin x + \sin x \ln|\sec x|$$

$$\boxed{y = C_1 \cos x + C_3 \sin x + x\cos x + \sin x \ln|\sec x|}$$

note: I'll also accept

$$\boxed{y = C_1 \cos x + C_2 \sin x + \cos x(x - \tan x) + \underbrace{\sin x \ln|\sec x|}_{\text{or } -\sin x \ln|\cos x|}}$$

3. (4 points) Solve the following DE for $x > 0$, given that $y(1) = 2, y'(1) = 6$.

$$4x^2y'' + 4x\,y' - y = 0$$

aux eqn: $4m(m-1) + 4m - 1 = 0$

$4m^2 - 4m + 4m - 1 = 0$

$4m^2 - 1 = 0$

$m = \pm \frac{1}{2}$ 1

$y = c_1 x^{\frac{1}{2}} + c_2 x^{-\frac{1}{2}}$ 1

now plug in initial conditions:

$y(1) = 2$

$2 = c_1 + c_2$

$y'(1) = 6$

$y' = \frac{1}{2} c_1 x^{-\frac{1}{2}} - \frac{1}{2} c_2 x^{-\frac{3}{2}}$

$6 = \frac{1}{2} c_1 - \frac{1}{2} c_2$

$12 = c_1 - c_2$

system is $\begin{cases} 2 = c_1 + c_2 \\ 12 = c_1 - c_2 \end{cases}$ 1

add eqns together

$14 = 2c_1$

$c_1 = 7$

$c_2 = -5$

$\boxed{y = 7x^{\frac{1}{2}} - 5x^{-\frac{1}{2}}}$ 1

4. (6 points) A mass of 1 kg is attached to an ideal spring with constant 13 N/m. There is a damping force such that the damping constant is numerically equal to 6. The mass is driven by an external force $F(t) = 5\sin t$, where $F(t)$ is in Newtons.

   (a) Find the equation of motion for the position $y(t)$ of the mass as a function of time. Assume that you do not know the initial conditions.

   (b) The mass is set in motion with some initial conditions and allowed to run on its own. After a long time has elapsed, you observe the motion of the mass. What does the motion look like? Choose all that apply.

      (i) A sine wave with exponentially decaying amplitude
      (ii) Beat frequency
      (iii) A sine wave
      (iv) The mass has stopped moving.
      (v) None of the above

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = F_{ext}(t)$$

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 13y = 5\sin t$$

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y_c: aux eqn $\quad r^2 + 6r + 13 = 0$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-6 \pm \sqrt{36 - 52}}{2} = \frac{-6 \pm 4i}{2}$$

$$= -3 \pm 2i$$

$$y_c = e^{-3t}(c_1 \cos 2t + c_2 \sin 2t) \qquad 1$$

y_p: let $\quad y_p = A\sin t + B\cos t \qquad 1$

$$y_p' = A\cos t - B\sin t$$

$$y_p'' = -A\sin t - B\cos t$$

plug back into DE: $\quad \frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 13y = 0$

$$(-A\sin t - B\cos t) + 6(A\cos t - B\sin t) + 13(A\sin t + B\cos t) = 5\sin t$$

1

$$(-A - 6B + 13A)\sin t + (-B + 6A + 13B)\cos t = 5\sin t$$

$$-A - 6B + 13A = 5$$

$$12A - 6B = 5$$

$$-B + 6A + 13B = 0$$

$$6A + 12B = 0$$

$$A = -2B$$

$$12(-2B) - 6B = 5$$

$$-30B = 5$$

$$B = -\tfrac{1}{6}$$

$$A = -2B = -2\left(-\tfrac{1}{6}\right) = \tfrac{1}{3}$$

1

$$y_p = \tfrac{1}{3}\sin t - \tfrac{1}{6}\cos t$$

1

1

$$\boxed{y = y_c + y_p = \underbrace{e^{-3t}(c_1\cos 2t + c_2\sin 2t)}_{\text{transient term, goes to zero as } t \text{ gets large}} + \underbrace{\tfrac{1}{3}\sin t - \tfrac{1}{6}\cos t}_{\text{sine wave with phase shift}}}$$