Math 252 – Quiz #1

April 21, 2011 **Name:** Solution Set
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Total: 25 points

1. Solve the following differential equation. Give an explicit solution (*i.e.*, solve for *m*). (5 points)

$$\frac{dm}{dt} = 4 - m^2 \quad \leftarrow \text{separable}$$

$$\frac{dm}{4-m^2} = dt \quad \leftarrow \text{partial fractions}$$

$$\frac{1}{(2-m)(2+m)} = \frac{A}{2-m} + \frac{B}{2+m}$$

$$1 = A(2+m) + B(2-m)$$

let $m = 2$

$$1 = 4A$$

let $m = -2$

$$1 = 4B$$

$$\therefore A = B = 1/4$$

$$\left[\frac{1}{4(2-m)} + \frac{1}{4(2+m)}\right] dm = dt$$

$$\left[\frac{1}{2-m} + \frac{1}{2+m}\right] dm = 4dt$$

$$-\ln|2-m| + \ln|2+m| = 4t + C$$

$$\ln\left|\frac{2+m}{2-m}\right| = 4t + C$$

$$\left|\frac{2+m}{2-m}\right| = e^{4t+C}$$

$$\frac{2+m}{2-m} = \underbrace{\pm e^{C}}_{= C_1} e^{4t}$$

$$2+m = (2-m)\, C_1 e^{4t}$$

$$m + mC_1 e^{4t} = 2C_1 e^{4t} - 2$$

$$\boxed{m = \frac{2(C_1 e^{4t} - 1)}{(C_1 e^{4t} + 1)}}$$

noting that it's separable: (1)

partial fractions: (1)

correct (1)

correct integration incl – sign (1)

solve for m (1)

2. Solve the following initial-value problem, giving an explicit solution. (5 points)

$$y' - 2xy = x \qquad \text{if } y = 2 \text{ when } x = 0$$

linear with integrating factor

$$e^{\int P(x)\,dx} = e^{\int -2x\,dx} = e^{-x^2} \quad ①$$

DE becomes

$$\frac{dy}{dx} e^{-x^2} - 2xy\,e^{-x^2} = x e^{-x^2}$$

$$\frac{d}{dx}\left(y e^{-x^2}\right) = x e^{-x^2}$$

$$y e^{-x^2} = \int x e^{-x^2}\,dx$$

①

let $u = -x^2$

$du = -2x\,dx$

$\frac{du}{-2} = x\,dx$

$$= \int \frac{e^u\,du}{-2}$$

$$= -\frac{1}{2} e^u + C \quad ①$$

$$y e^{-x^2} = -\tfrac{1}{2} e^{-x^2} + C$$

$$y = -\tfrac{1}{2} + C e^{x^2}$$

when $x = 0$, $y = 2 = -\tfrac{1}{2} + C$

$C = \tfrac{5}{2}$

①

so $\boxed{y = \frac{5}{2} e^{x^2} - \frac{1}{2}}$ ①

3. Solve the following differential equation. Give an explicit solution. (5 points)

$$\frac{dy}{dx} = \sin^2(x-y)$$

Let $u = x - y$ (1/2)

$$\frac{du}{dx} = 1 - \frac{dy}{dx}$$

$$\frac{dy}{dx} = 1 - \frac{du}{dx}$$ (1)

so DE becomes

$$1 - \frac{du}{dx} = \sin^2 u$$

$$1 - \sin^2 u = \frac{du}{dx}$$

separable $\rightarrow$ $dx = \frac{du}{1 - \sin^2 u}$ (1)

$$dx = \frac{du}{\cos^2 u}$$ (1/2)

$$dx = \sec^2 u \, du$$

$$x + C = \tan u$$

$$x + C = \tan(x-y)$$ (1)

$$\tan^{-1}(x+C) = x - y$$

$$\boxed{y = x - \tan^{-1}(x+C)}$$ (1)

4. Solve the following Bernoulli differential equation. (5 points)

$$x\,y' + y = \frac{\sin x}{xy^2}$$

Standard form:

$$y' + \frac{1}{x}y = y^{-2}\frac{\sin x}{x^2}$$

Bernoulli with $n = -2$

$$u = y^{1-n} = y^3 \quad \text{½}$$

$$\frac{du}{dx} = 3y^2\frac{dy}{dx} \quad \text{1}$$

multiplying both sides by $3y^2$

$$\underbrace{3y^2\frac{dy}{dx}}_{\frac{du}{dx}} + \frac{3}{x}\underbrace{y^3}_{u} = \frac{3\sin x}{x^2} \quad \text{½}$$

$$\frac{du}{dx} + \frac{3}{x}u = \frac{3}{x^2}\sin x \quad \text{1}$$

$\leftarrow$ linear with integrating factor

$$e^{\int P(x)\,dx} = e^{\int \frac{3}{x}dx} = e^{3\ln|x|} = x^3 \quad \text{1}$$

$$\frac{du}{dx}x^3 + 3x^2u = 3x\sin x$$

$$\frac{d}{dx}(ux^3) = 3x\sin x$$

$$ux^3 = \int 3x\sin x\,dx$$

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$3x$	$+$	$\sin x$
3	$-$	$-\cos x$
0		$-\sin x$

$$\boxed{y^3x^3 = -3x\cos x + 3\sin x + C} \quad \text{1}$$

or

$$\boxed{y^3 = -\frac{3}{x^2}\cos x + \frac{3}{x^3}\sin x + \frac{C}{x^3}}$$

or, if you insist

$$y = \left(-\frac{3}{x^2}\cos x + \frac{3}{x^3}\sin x + \frac{C}{x^3}\right)^{1/3} = \frac{1}{x}\sqrt[3]{-3x\cos x + 3\sin x + C}$$

etc, ...

5. Find the solution to the following equation. (5 points)

$$(x+\sin 2y)\,dx+(2x\cos 2y-2y)\,dy=0$$

it's exact if

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$; $\frac{\partial M}{\partial y} = 2\cos 2y$ $\frac{\partial N}{\partial x} = 2\cos 2y$

equal, so DE is exact 1

so $M = \frac{\partial f}{\partial x} = x + 2\sin 2y$

$f(x,y) = \int M\,dx = \int (x + \sin 2y)\,dx$

$= \frac{x^2}{2} + x\sin 2y + g(y)$ 1

$\frac{\partial f}{\partial y} = 2x\cos 2y + g'(y)$ but this = N 1

$= 2x\cos 2y - 2y$

$\therefore$ $g'(y) = -2y$

$g(y) = -y^2$ 1

and $f(x,y) = \frac{x^2}{2} + x\sin 2y - y^2$

$\therefore$ $\boxed{\frac{x^2}{2} + x\sin 2y - y^2 = C}$ 1