Math 252 – Quiz #2

May 9, 2011 **Name:** Solution Set
Instructor: Patricia Wrean

Total: 25 points

1. Solve the following initial value problem, given that $y(0) = 3$ and $y'(0) = 32$. (3 points)

$$y'' + 2y' - 24y = 0$$

aux eqn: $m^2 + 2m - 24 = 0$

$$(m+6)(m-4) = 0$$

$$m = -6, 4$$

$$y = C_1 e^{-6x} + C_2 e^{4x}$$ (1)

IVP:

at $x = 0$, $y = 3$

$$3 = C_1 + C_2$$

at $x = 0$, $y' = 32$

$$y' = -6C_1 e^{-6x} + 4C_2 e^{4x}$$

$$32 = -6C_1 + 4C_2$$

(1)

$$\therefore \left.\begin{array}{l} C_1 + C_2 = 3 \\ -3C_1 + 2C_2 = 16 \end{array}\right\} \quad C_1 = -2, \; C_2 = 5$$

$$\boxed{y = -2e^{-6x} + 5e^{4x}}$$ (1)

2. Consider the following differential equations and graphs. (6 points)

a) For the following DE, give the general solution.

$$y'' + 12y' + 40y = 0$$

$m^2 + 12m + 40 = 0$

$m = \dfrac{-12 \pm \sqrt{144-160}}{2}$

$= -6 \pm 2i$

$y = e^{-6x}(C_1 \cos 2x + C_2 \sin 2x)$

1

b) For the following DE, first state the complementary solution to the associated homogeneous DE. Then give the form of the particular solution to the DE, leaving your answer with undetermined coefficients.

$$y'' - 4y = 3e^{-4x}$$

$m^2 - 4 = 0$

$m = \pm 2$

$\boxed{y_c = C_1 e^{2x} + C_2 e^{-2x}}$

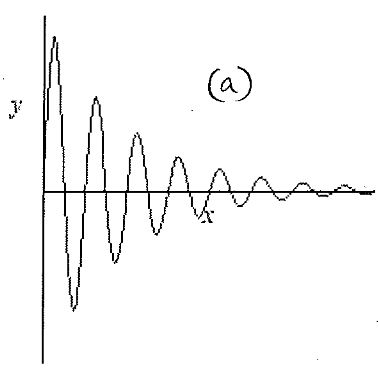
1

$\boxed{y_p = Ae^{-4x}}$ not the "bad case"

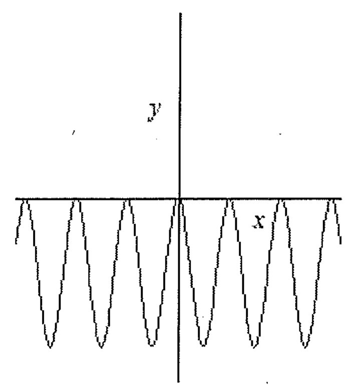
1

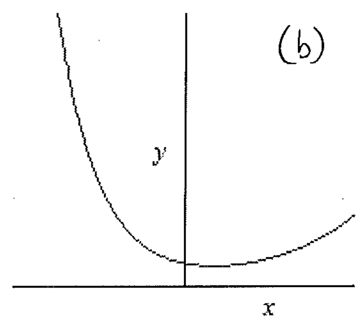
c) For each DE above, indicate which plot or plots below are possible graphs of the general solution to that DE. No explanation is required. You may pick more than one.

3

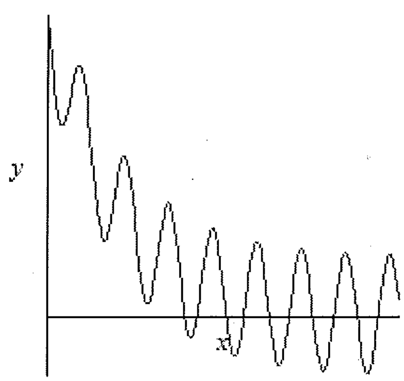


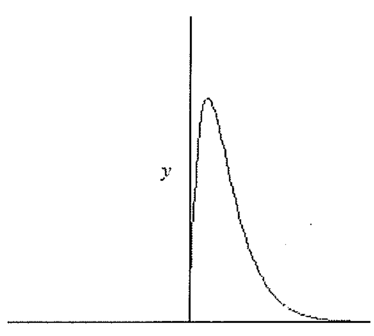
(a)

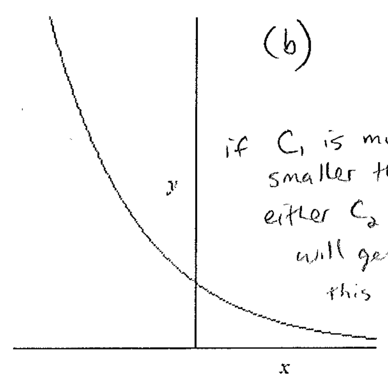




(b)







(b)

if C_1 is much smaller than either C_2 or A, will get this

3. Consider the following DE: (5 points)

$$(x-1)y'' - xy' + y = 0$$

a) Verify that $y_1 = e^x$ is a solution to this DE
b) Find a second solution y_2, and give the general solution y for this DE.

(Hint: $\frac{x}{x-1} = 1 + \frac{1}{x-1}$.)

a)

$y_1 = e^x$
$y_1' = e^x$
$y_1'' = e^x$

$(x-1)e^x - xe^x + e^x = 0$
$xe^x - e^x - xe^x + e^x = 0$
$0 = 0$ ✓ 1

b)

$$y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{y_1^2} dx$$

where $P(x) = -\frac{x}{x-1}$ from std form of eqn 1

$$= e^x \int \frac{e^{\int (1 + \frac{1}{x-1})dx}}{e^{2x}}$$

$$= e^x \int \frac{e^{x + \ln|x-1|}}{e^{2x}} dx \quad 1$$

$$= e^x \int \frac{(x-1)e^x}{e^{2x}} dx$$

$$= e^x \int (x-1)e^{-x} dx$$

D		I
$x-1$	+	e^{-x}
1	−	$-e^{-x}$
0		e^{-x}

quick check of y_2:
$y_2' = -1$
$y_2'' = 0$
so $-x(-1) + (-x) = 0$
✓

$$= e^x \left[-(x-1)e^{-x} - e^{-x}\right]$$

$$= -(x-1) - 1$$

$\boxed{y_2 = -x}$ 1 and $\boxed{y = C_1 e^x + C_2 x}$ 1

4. Solve the following differential equation. (5 points)

$$y'' - 2y' + 2y = x + 10\sin 2x$$

complementary:

$m^2 - 2m + 2 = 0$

$m = \dfrac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$

$y_c = e^x (C_1 \cos x + C_2 \sin x)$ 1

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particular

$y_p = Ax + B + C\sin 2x + D\cos 2x$ 1 ← not the "bad case"

$y_p' = A + 2C\cos 2x - 2D\sin 2x$

$y_p'' = -4C\sin 2x - 4D\cos 2x$

$y'' - 2y' + 2y = x + 10\sin 2x$ 1

$(-4C\sin 2x - 4D\cos 2x) - 2(A + 2C\cos 2x - 2D\sin 2x) + 2(Ax + B + C\sin 2x + D\cos 2x) = x + 10\sin 2x$

$-4C\sin 2x - 4D\cos 2x - 2A - 4C\cos 2x + 4D\sin 2x + 2Ax + 2B + 2C\sin 2x + 2D\cos 2x = x + 10\sin 2x$

$(-4C + 4D + 2C)\sin 2x + (-4D - 4C + 2D)\cos 2x + 2Ax + 2B - 2A = x + 10\sin 2x$

so $-4C + 4D + 2C = 10$ and $-4D - 4C + 2D = 0$ from sin & cos

$4D - 2C = 10$

$2D - C = 5$ $\xleftrightarrow{2\times 2 \text{ system}}$ $-2D - 4C = 0$

$-4C = 2D$

$2D = C + 5$ → $C + 5 = -4C$ ←

$\boxed{C = -1}$ and $\boxed{D = 2}$ 1

also $2A = 1$ so $\boxed{A = \tfrac{1}{2}}$

$2B - 2A = 0$ so $\boxed{B = \tfrac{1}{2}}$

so $\boxed{y_p = \tfrac{1}{2}x + \tfrac{1}{2} - \sin 2x + 2\cos 2x}$

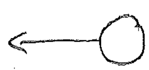
$$\boxed{y = y_c + y_p = e^x(C_1\cos x + C_2 \sin x) + \tfrac{1}{2}(x+1) - \sin 2x + 2\cos 2x}$$

1

5. An object moving through a fluid has a drag force acting on it which is proportional to its speed. Assume that the drag force is the only unbalanced force acting on the object. If the initial speed of the object is v_o, (6 points)

a) find the object's speed as a function of time t
b) find the magnitude of the object's displacement x as a function of time t
c) calculate $\lim_{t\to\infty} x(t)$. In real life, what does this limit correspond to?

a) free body diagram:



---> direction of motion

where $F_{drag} \propto v$

$F_{drag} = -kv$

Newton's 2nd Law:

$$\sum \vec{F} = m\vec{a}$$

$$F_{drag} = ma$$

$$\boxed{-kv = m\frac{dv}{dt}}$$ (1)

So the DE is $\dfrac{dv}{dt} = -\dfrac{k}{m}v$ ← 1st order linear & is separable

$$\frac{dv}{v} = -\frac{k}{m}dt$$

$$\ln|v| = -\frac{k}{m}t + C$$

$$|v| = e^{-\frac{k}{m}t + C}$$

$$v = (\pm e^{C})e^{-\frac{k}{m}t}$$

$$v = C_1 e^{-\frac{k}{m}t}$$

initial value: at $t=0$, $v = v_0$ so $v_0 = C_1 e^0$

$C_1 = v_0$

and $\boxed{v = v_0 e^{-\frac{k}{m}t}}$ (1)

b) method #1:

$$x = \int v\,dt$$

$$= \int v_0 e^{-k/m\,t}\,dt$$

$$= -\frac{mv_0}{k}e^{-k/m\,t} + C$$ (1)

or

method #2:

$$v = v_0 e^{-kt/m}$$

$$\frac{dx}{dt} = v_0 e^{-kt/m}$$

$$dx = v_0 e^{-kt/m}dt$$ ← integrate both sides & get same as

initial value: at $t=0$, $x=0$

$$0 = -\frac{mv_0}{k} + C \qquad \text{so} \qquad C = \frac{mv_0}{k}$$

$$x = \frac{mv_0}{k} - \frac{mv_0}{k} e^{-\frac{k}{m}t}$$

$$\boxed{x = \frac{mv_0}{k}\left(1 - e^{-\frac{k}{m}t}\right)} \qquad 1$$

c) as $t \to \infty$, $e^{-\frac{k}{m}t} \to 0$ and 1

$x \to \dfrac{mv_0}{k}$, a finite limit

This limit is then the total distance that the object travels through the liquid. 1