Math 252 – Quiz #3

May 27, 2011 **Name:** Solution Set
Instructor: Patricia Wrean

Total: 20 points

1. A mass of 0.50 kg is attached to a spring of constant 8.0 N/m. Assume that there is no damping. The mass is released 6.0 cm below equilibrium with an upward velocity of 0.12 m/s. (~~5~~ 6 points)

   a) Find the position of the mass as a function of time.
   b) State the period and amplitude of this motion.
   c) What is the magnitude of the instantaneous velocity of the mass when it passes through the equilibrium position?

a) $m\frac{d^2x}{dt^2} + \cancel{b\frac{dx}{dt}}^{\,0} + kx = 0$ (no external forces)

$0.5\frac{d^2x}{dt^2} + 8x = 0$ has aux eqn

$0.5r^2 + 8 = 0$

$r^2 = -16$

$r = \pm 4i$

so $x = C_1 \cos 4t + C_2 \sin 4t$ 1

initial conditions: $x(0) = -0.06 = C_1$

$\dot{x}(0) = 0.12 = -4C_1 \cancel{\sin 4t}^{\,0} + 4C_2 \cancel{\cos 4t}^{\,1}$

so $4C_2 = 0.12$ and $C_2 = 0.03$

$\boxed{x = -0.06 \cos 4t + 0.03 \sin 4t}$ 1

b) $x = A\sin(\omega t + \phi)$ where $A = \sqrt{C_1^2 + C_2^2} = \sqrt{(-0.06)^2 + (0.03)^2}$

$= 0.067 \sin(4t + \phi)$ $= 0.067$ m

so amplitude is $\boxed{0.067 \text{ m}}$ 1 and period is $\frac{2\pi}{4} = \boxed{\pi/2 \text{ seconds}}$ 1

c) $x = A\sin(\omega t + \phi)$ ← if mass is at $x = 0$, then $\sin(\omega t + \phi) = 0$

$\dot{x} = \omega A \cos(\omega t + \phi)$ and $\cos(\omega t + \phi) = \pm 1$

so magnitude of velocity $= \omega A = 4(0.067) = \boxed{0.27 \text{ m/s}}$ 2 if you insist, for part (c) you can calculate the phase shift.

$$A \sin(\omega t + \phi) = A \sin \omega t \cos\phi + A \cos \omega t \sin\phi$$

$\uparrow$

so $A\cos\phi$ = coeff on $\sin \omega t = 0.03$

and $\nearrow$ $A \sin\phi$ = coeff on $\cos \omega t = -0.06$

so $\cos\phi$ is $+$
and $\sin\phi$ is $-$ $\Big\}$ Q IV and

$$\phi = \arctan\left(\frac{A\sin\phi}{A\cos\phi}\right)$$

$$= \arctan\left(-\frac{0.06}{0.03}\right)$$

$$= -1.11 \text{ rads}$$

note: if used opposite sign convention, will get $+2.03$ instead

so $x = A\sin(\omega t + \phi)$

$= 0.067 \sin(4t - 1.11 \text{ rads})$

for what t does $\sin(4t - 1.11) = 0$?

when $4t - 1.11 = 0 + n\pi$

$$t = \frac{1.11 + n\pi}{4}$$

$= +0.28 + \frac{n}{4}\pi$ for $n = 1, 2, 3, \ldots$

at these times, $\cos(4t - 1.11) = \cos(1.11 + n\pi - 1.11)$

$= \cos(n\pi)$

$= \pm 1$

(now, wasn't that a longwinded way to find that when $\sin\theta = 0$, $\cos\theta = \pm 1$?)

etc

2. Solve the following DE. 7 (~~5~~ points)

$$x^2 y'' - 3xy' + 4y = \frac{1}{x}$$

Hint: use integration by parts for $\int x^n \ln x \, dx$.

<u>Complementary soln:</u> LHS is Cauchy-Euler

aux eqn: $am^2 + (b-a)m + c = 0$

$m^2 - 4m + 4 = 0$ 1

$(m-2)^2 = 0$

$m = 2$ (repeated roots)

$y_c = \underbrace{C_1 x^2}_{y_1} + \underbrace{C_2 x^2 \ln x}_{y_2}$ 1

particular soln: variation of parameters with $f(x) = \frac{1}{x^3}$ 1

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x^2 & x^2\ln x \\ 2x & x + 2x\ln x \end{vmatrix} = x^3 + 2x^3\ln x - 2x^3 \ln x = \underline{\underline{x^3}}$$

$$W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix} = \begin{vmatrix} 0 & x^2\ln x \\ 1/x^3 & x + 2x\ln x \end{vmatrix} = \frac{-\ln x}{x}$$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix} = \begin{vmatrix} x^2 & 0 \\ 2x & \frac{1}{x^3} \end{vmatrix} = \frac{1}{x}$$

1

so $u_1 = \int \frac{W_1}{W} dx = \int \frac{-\ln x}{x^4} dx = -\left[-\frac{1}{3x^3}\ln x + \int \frac{1}{3x^4} dx \right]$

$= -\left[-\frac{\ln x}{3x^3} - \frac{1}{9x^3} \right]$

$= \frac{\ln x}{3x^3} + \frac{1}{9x^3}$ 1

D	I
$\ln x$	$\frac{1}{x^4}$
$\frac{1}{x}$	$-\frac{1}{3x^3}$

$$u_2 = \int \frac{W_2}{W}\,dx = \int \frac{1}{x^4}\,dx = -\frac{1}{3x^3}$$

so $y_p = u_1 y_1 + u_2 y_2$

$$= \left(\frac{\ln x}{3x^3} + \frac{1}{9x^3}\right)x^2 + \left(-\frac{1}{3x^3}\right)x^2 \ln x$$

$$= \cancel{\frac{\ln x}{3x}} + \frac{1}{9x} - \cancel{\frac{\ln x}{3x}}$$

$$= \frac{1}{9x} \qquad 1$$

$$y = y_c + y_p$$

$$\boxed{y = C_1 x^2 + C_2 x^2 \ln x + \frac{1}{9x}} \qquad 1$$

note: if screwed up and thought $f(x) = \frac{1}{x}$ instead of $\frac{1}{x^3}$, should get

$y_p = x$

$y = C_1 x^2 + C_2 x^2 \ln x + x$ instead

if screwed up W so integration not possible -3

($-2\frac{1}{2}$ if wrote form of final answer)

3. Solve the following differential equation, given that $y(0) = -4$ and $y'(0) = 5$. Give the first five non-zero terms of the solution. ($\cancel{5}$ points) 6

$$y'' + (x^2 - 1)y = 0$$

let $y = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \dots$

$y' = C_1 + 2C_2 x + 3C_3 x^2 + \dots$

$y'' = 2C_2 + 3\cdot 2 C_3 x + 4\cdot 3 C_4 x^2 + \dots$ 1

sub back into DE:

$$y'' + (x^2 - 1)y = 0$$

$(2C_2 + 6C_3 x + 12 C_4 x^2 + \dots) + (x^2 - 1)(C_0 + C_1 x + C_2 x^2 + \dots) = 0.$ 1

$(2C_2 - C_0) + (6C_3 - C_1)x + (12C_4 + C_0 - C_2)x^2 + \dots = 0$ 1

so want first 5 non zero terms

$C_0 = y(0) = -4$

$C_1 = y'(0) = 5$

$C_2 = \frac{C_0}{2} = -2$

$C_3 = \frac{1}{6} C_1 = 5/6$

$C_4 = \frac{1}{12}(C_2 - C_0) = \frac{1}{12}(-2 + 4) = \frac{1}{6}$

2

and $\boxed{y = -4 + 5x - 2x^2 + \frac{5}{6}x^3 + \frac{1}{6}x^4 + \dots}$ 1 if you insist, can find the reccurrence relation instead:

let $y = \sum_{n=0}^{\infty} C_n x^n$

$y' = \sum_{n=1}^{\infty} n C_n x^{n-1}$

$y'' = \sum_{n=2}^{\infty} n(n-1) C_n x^{n-2}$

sub into DE:

$$\sum_{n=2}^{\infty} n(n-1) C_n x^{n-2} + (x^2-1) \sum_{n=0}^{\infty} c_n x^n = 0 \qquad (1)$$

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=0}^{\infty} C_n x^{n+2} - \sum_{n=0}^{\infty} c_n x^n = 0$$

↑ lowest is x^0 ↑ lowest is x^2 ↑ lowest is x^0

$$2C_2 + 6C_3 x + \sum_{n=4}^{\infty} n(n-1) C_n x^{n-2} + \sum_{n=0}^{\infty} C_n x^{n+2} - C_0 - C_1 x - \sum_{n=2}^{\infty} C_n x^n = 0 \qquad (1)$$

$$(2C_2 - C_0) + (6C_3 - C_1)x + \sum_{n=4}^{\infty} n(n-1) C_n x^{n-2} + \sum_{n=0}^{\infty} C_n x^{n+2} - \sum_{n=2}^{\infty} C_n x^n = 0$$

↑ let $k = n-2$, $k+2 = n$ ↑ let $k = n+2$, $k-2 = n$ ↑ let $k = n$

$$(2C_2 - C_0) + (6C_3 - C_1)x + \sum_{k=2}^{\infty} (k+2)(k+1) C_{k+2} x^k + \sum_{k=2}^{\infty} C_{k-2} x^k - \sum_{k=2}^{\infty} C_k x^k = 0$$

$$(2C_2 - C_0) + (6C_3 - C_1)x + \sum_{k=2}^{\infty} \left[(k+2)(k+1) C_{k+2} + C_{k-2} - C_k \right] x^k = 0 \qquad (1)$$

so $C_2 = \frac{1}{2} C_0$

$C_3 = \frac{1}{6} C_1$

and $(k+2)(k+1)c_{k+2} + c_{k-2} - c_k = 0$ for $k = 2, 3, 4, \ldots$

$$c_{k+2} = \frac{c_k - c_{k-2}}{(k+2)(k+1)} \quad (1)$$

so

$$\left.\begin{aligned} c_0 &= y(0) = -4 \\ c_1 &= y'(0) = 5 \\ c_2 &= \tfrac{1}{2} c_0 = -2 \\ c_3 &= \tfrac{1}{6} c_1 = 5/6 \\ k=2 \quad c_4 &= \frac{c_2 - c_0}{12} = \frac{-2+4}{12} = \frac{1}{6} \end{aligned}\right\} \quad (1)$$

$$\boxed{y = -4 + 5x - 2x^2 + \frac{5}{6}x^3 + \frac{1}{6}x^4 + \ldots} \quad (1)$$

blech!

if got to c_{k+2} but no further
or screwed up inital conds
(2)

4. Consider the power series solution to the following DE about the ordinary point $x = 0$.
(~~8~~ points)
6

$$(x+1)y'' + xy = 0$$

a) For which values of x can we guarantee that the series converge?

b) Find the recurrence relation for the coefficients of the series. Do not bother to calculate any coefficients.

a) singular point at $x=-1$ so interval is $\boxed{|x|<1}$ 1

b) let $y = \sum_{n=0}^{\infty} C_n x^n$

$y' = \sum_{n=1}^{\infty} n C_n x^{n-1}$

$y'' = \sum_{n=2}^{\infty} n(n-1) C_n x^{n-2}$ 1

sub into DE:

$$(x+1)y'' + xy = 0$$

$$(x+1)\sum_{n=2}^{\infty} n(n-1)C_n x^{n-2} + x\sum_{n=0}^{\infty} C_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)C_n x^{n-1} + \sum_{n=2}^{\infty} n(n-1)C_n x^{n-2} + \sum_{n=0}^{\infty} C_n x^{n+1} = 0$$ 1

lowest is x^1 lowest is x^0 lowest is x^1

$$\sum_{n=2}^{\infty} n(n-1)C_n x^{n-1} + 2C_2 + \sum_{n=3}^{\infty} n(n-1)C_n x^{n-2} + \sum_{n=0}^{\infty} C_n x^{n+1} = 0$$ 1

let $k=n-1$, $k+1=n$ let $k=n-2$, $k+2=n$ let $k=n+1$, $k-1=n$

$$\sum_{k=1}^{\infty} (k+1)k C_{k+1} x^k + 2C_2 + \sum_{k=1}^{\infty} (k+2)(k+1)C_{k+2} x^k + \sum_{k=1}^{\infty} C_{k-1} x^k = 0$$

so $\quad 2c_2 + \sum_{k=1}^{\infty} \left[k(k+1)\, c_{k+1} + (k+1)(k+2)\, c_{k+2} + c_{k-1} \right] x^k = 0 \qquad ①$

recurrence relation:

$$(k+1)(k+2)\, c_{k+2} = -\, k(k+1)\, c_{k+1} - c_{k-1}$$

for $k = 1, 2, 3, \ldots$

$$\boxed{c_{k+2} = -\frac{k}{k+2}\, c_{k+1} - \frac{c_{k-1}}{(k+1)(k+2)}} \qquad ①$$

or $\boxed{= \dfrac{-k(k+1)\, c_{k+1} - c_{k-1}}{(k+1)(k+2)}}$