Math 252 – Quiz #4

June 10, 2011 **Name:** Solution Set.
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Total: 25 points

1. Evaluate the following. ($\cancel{12}$ 4 points)

a) $\mathcal{L}\left\{t^3e^{7t}\right\}$

$\mathcal{L}\{t^3\} = \dfrac{3!}{s^4}$ and e^{7t} makes $s \to s-7$

so $\boxed{\mathcal{L}\{t^3e^{7t}\} = \dfrac{3!}{(s-7)^4}}$

(2)

b) $\mathcal{L}^{-1}\left\{\dfrac{e^{-2s}}{s-3}\right\}$

$\mathcal{L}^{-1}\left\{\dfrac{1}{s-3}\right\} = e^{3t}\,\mathcal{L}^{-1}\left\{\dfrac{1}{s}\right\} = e^{3t}$

$\boxed{\mathcal{L}^{-1}\left\{\dfrac{e^{-2s}}{s-3}\right\} = e^{3(t-2)}\,\mathcal{U}(t-2)}$

(2)

2. Evaluate. (5 points)

a) $\mathcal{L}\left\{t\left(1-e^{3t}\right)^2\right\} = \mathcal{L}\left\{t\left(1-2e^{3t}+e^{6t}\right)\right\}$

$$\boxed{= \frac{1}{s^2} - \frac{2}{(s-3)^2} + \frac{1}{(s-6)^2}}$$

2

if get $\frac{1}{s}\left(\frac{1}{s} - \frac{2}{s-3} + \frac{1}{s-6}\right)$, -1

$(-1)\frac{d}{ds}F(s)$, -2

b) $\mathcal{L}\left\{te^{2t}\cos 4t\right\}$

$$\mathcal{L}\{t\cos 4t\} = \frac{s^2-16}{(s^2+16)^2}$$

so $\boxed{\mathcal{L}\{e^{2t}t\cos 4t\} = \frac{(s-2)^2-16}{((s-2)^2+16)^2}}$

3

if you insist,

$$= \frac{s^2-4s-12}{(s^2-4s+20)^2}$$

3. Evaluate. (4 points)

$$\mathcal{L}^{-1}\left\{\frac{s}{(s+3)(s^2+9)}\right\}$$

partial fractions:

$$\frac{s}{(s+3)(s^2+9)} = \frac{A}{s+3} + \frac{Bs+C}{s^2+9}$$

coverup: $A = \frac{-3}{18} = -\frac{1}{6}$.

could also say

$s = A(s^2+9) + (Bs+C)(s+3)$

when $s=0$

$0 = 9A + 3C$ so $C = -3A = \frac{1}{2}$

when $s=1$

$1 = 10A + 4B + 4C$

$4B = 1 + \frac{10}{6} - 2 = \frac{4}{6}$ so $B = \frac{1}{6}$

$$B(3i) + C = \frac{3i}{3i+3} = \frac{i}{i+1}\left(\frac{i-1}{i-1}\right)$$

$$= \frac{i^2 - i}{i^2 - 1}$$

$$3Bi + C = -\frac{1-i}{-2} = \frac{i}{2} + \frac{1}{2}$$

so $3Bi = \frac{i}{2}$ and $C = \frac{1}{2}$

$B = \frac{1}{6}$

so

$$\mathcal{L}^{-1}\left\{\frac{s}{(s+3)(s^2+9)}\right\} = \mathcal{L}^{-1}\left\{-\frac{1}{6}\cdot\frac{1}{s+3} + \frac{1}{6}\frac{s}{s^2+9} + \frac{1}{2}\frac{1}{s^2+9}\right\}$$

$$= -\frac{1}{6}e^{-3t} + \frac{1}{6}\cos 3t + \mathcal{L}^{-1}\left\{\frac{1}{6}\frac{3}{s^2+9}\right\}$$

$$\boxed{= -\frac{1}{6}e^{-3t} + \frac{1}{6}\cos 3t + \frac{1}{6}\sin 3t}$$

4. Use Laplace transforms to solve the following IVP. (~~6~~ 5 points)

$y'' + 4y = \cos 2t + 4\sin 2t$ for $y(0) = 1$ and $y'(0) = 6$

$$s^2 Y(s) - s y(0) - y'(0) + 4Y(s) = \frac{s}{s^2+4} + 4 \cdot \frac{2}{s^2+4}$$

$$(s^2+4) Y(s) - s - 6 = \text{''}$$

$$Y(s) = \frac{s}{s^2+4} + \frac{6}{s^2+4} + \frac{s}{(s^2+4)^2} + \frac{8}{(s^2+4)^2}$$

$$= \frac{s}{s^2+4} + 3 \cdot \frac{2}{s^2+4} + \frac{1}{4}\frac{4s}{(s^2+4)^2} + 8 \frac{1}{(s^2+4)^2}$$

$$y(t) = \cos 2t + 3\sin 2t + \frac{1}{4} t \sin 2t + 8\left(\frac{\sin 2t - 2t\cos 2t}{16}\right)$$

$$= \cos 2t + 3\sin 2t + \frac{1}{4} t\sin 2t + \frac{1}{2}\sin 2t - t\cos 2t$$

$$\boxed{y(t) = \cos 2t + \frac{7}{2}\sin 2t - t\cos 2t + \frac{1}{4} t \sin 2t}$$

5. Consider the following IVP, where $y(0)=0$. (7 points)

$$y'+3y=f(t)\text{, where } f(t)=\begin{cases}12, & 0\le t<2\\ 0, & 2\le t<4\\ 12, & t\ge 4\end{cases}$$

a) Solve this DE.
b) Calculate the value of $y(t)$ for $t=3$.

a) $sY(s) - y'(0) + 3Y(s) = \mathcal{L}\{f(t)\}$

now $f(t) = 12\left[1-\mathcal{U}(t-2)\right] + 12\,\mathcal{U}(t-4)$

$= 12 - 12\,\mathcal{U}(t-2) + 12\,\mathcal{U}(t-4)$

$\mathcal{L}\{f(t)\} = \dfrac{12}{s} - \dfrac{12e^{-2s}}{s} + \dfrac{12e^{-4s}}{s}$

so $(s+3)Y(s) = \dfrac{12}{s}\left[1-e^{-2s}+e^{-4s}\right]$

$Y(s) = \dfrac{12}{s(s+3)}\left[1-e^{-2s}+e^{-4s}\right]$

partial fractions:

$\dfrac{12}{s(s+3)} = \dfrac{A}{s} + \dfrac{B}{s+3}$ where, using coverup, $A = \dfrac{12}{3} = 4$ and $B = \dfrac{12}{-3} = -4$

$Y(s) = \left(\dfrac{4}{s} - \dfrac{4}{s-3}\right)\left[1-e^{-2s}+e^{-4s}\right]$

now $\mathcal{L}^{-1}\left\{\dfrac{4}{s} - \dfrac{4}{s-3}\right\} = 4-4e^{-3t}$

so $\boxed{y(t) = 4-4e^{-3t} - \left(4-4e^{-3(t-2)}\right)\mathcal{U}(t-2) + \left(4-4e^{-3(t-4)}\right)\mathcal{U}(t-4)}$

b) at $t=3$ $\quad y(3) = 4-4e^{-3(3)} - \left(4-4e^{-3}\right)(1) + \cancel{\left(4-4e^{-3(-1)}\right)}^{\,0}(0)$

so $\boxed{y(3) = 4e^{-3} - 4e^{-9}}$ ≈ 0.198655 if you insist

≈ 0.20