Math 252 – Test 1

February 6, 2020 **Name:** Solution Set
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Total: 25 points

1. (5 points) Solve the following DE, giving an explicit solution.

$$\sin 3x dx + 2y^2 \cos^3 3x dy = 0$$

$$2y^2 \cos^3 3x \, dy = -\sin 3x \, dx$$

$$2y^2 dy = -\frac{\sin 3x}{\cos^3 3x} dx \qquad \text{(separable)}$$

$$\int 2y^2 dy = \int -\frac{\sin 3x}{\cos^3 3x} dx$$

$$\frac{2}{3} y^3 = -\frac{1}{6} \frac{1}{\cos^2 3x} + C \quad *$$

$$y^3 = -\frac{1}{4} \frac{1}{\cos^2 3x} + \frac{3}{2}C \quad C_1$$

$$\boxed{y = \sqrt[3]{C_1 - \frac{1}{4}\sec^2 3x}} \quad \text{or} \quad \boxed{y = \sqrt[3]{C_1 - \frac{1}{4}\tan^2 3x}}$$

* this integral can be done as:

let $u = \cos 3x$

$du = -3 \sin 3x \, dx$

$$\int -\frac{\sin 3x}{\cos^3 3x} dx = \int \frac{1}{3} \frac{du}{u^3}$$

$$= -\frac{1}{6} u^{-2} + C$$

$$= -\frac{1}{6} \frac{1}{\cos^2 3x} + C$$

or rewrite using trig:

$$\int -\frac{\sin 3x}{\cos^3 3x} dx = -\int \frac{\sin 3x}{\cos 3x} \cdot \frac{1}{\cos^2 3x} dx$$

$$= -\int \underbrace{\tan 3x}_{u} \underbrace{\sec^2 3x \, dx}_{du/3}$$

one of
$$\begin{cases} -\frac{1}{6}\tan^2 3x + C \\ -\frac{1}{6}\sec^2 3x + C \end{cases}$$

if you insist, could also do:

$$\underbrace{\sin 3x}_{M}\,dx + \underbrace{2y^2\cos^3 3x}_{N}\,dy = 0$$

$$\frac{\partial M}{\partial y} = 0 \qquad \frac{\partial N}{\partial x} = -18\,y^2\cos^2 3x \sin 3x$$

$$\frac{M_y - N_x}{N} = \frac{18y^2\cos^2 3x\sin 3x}{2y^2\cos^3 3x} = 9\,\frac{\sin 3x}{\cos 3x} = 9\tan 3x \quad \leftarrow f(x) \text{ only}$$

$$IF = e^{\int \frac{M_y - N_x}{N}dx} = e^{\int 9\tan 3x\,dx} = e^{3\ln|\sec 3x|}$$

$$= |\sec^3 3x| = \sec^3 x$$

then DE becomes

$$\sin 3x\sec^3 3x\,dx + 2y^2\cos^3 3x\sec^3 3x\,dy = 0$$

$$\tan 3x\sec^2 3x\,dx + 2y^2dy = 0$$

$$\frac{\partial M}{\partial y} = 0 = \frac{\partial N}{\partial x} \qquad \text{exact}$$

so $f(x,y) = C$

$$f(x,y) = \int \tan 3x\sec^2 3x\,dx = \frac{1}{6}\tan^2 3x + g(y)$$

and $$= \int 2y^2dy = \frac{2}{3}y^3 + h(x)$$

$$\text{so}\quad f(x,y) = C = \frac{1}{6}\tan^2 3x + \frac{2}{3}y^3$$

$$\text{so}\quad \frac{2}{3}y^3 = C - \frac{1}{6}\tan^2 3x$$

$$y^3 = \frac{3}{2}C - \frac{1}{4}\tan^2 3x$$

$$= C_1 - \frac{1}{4}\tan^2 3x$$

$$y = \sqrt[3]{C_1 - \frac{1}{4}\tan^2 3x} \qquad \text{as before}$$

2. (5 points) Consider the following initial-value problem.

$$xy' - 2y = x^4 - \frac{1}{x^4}, \qquad y(-1) = 2$$

(a) Solve this DE.

$\frac{dy}{dx} - \frac{2}{x} y = x^3 - \frac{1}{x^5}$ ← linear, 1st order, $P(x) = -\frac{2}{x}$

$IF = e^{\int P(x)dx}$
$= e^{\int -\frac{2}{x} dx}$
$= e^{-2\ln|x|} = e^{\ln|x|^{-2}}$
$= \frac{1}{|x|^2} = \frac{1}{x^2}$

then

$\frac{1}{x^2}\frac{dy}{dx} - \frac{2}{x^3} y = \frac{1}{x^2}\left(x^3 - \frac{1}{x^5}\right)$

$\int \frac{d}{dx}\left(y \cdot \frac{1}{x^2}\right)dx = \int \left(x - \frac{1}{x^7}\right) dx$

$y \cdot \frac{1}{x^2} = \frac{1}{2}x^2 + \frac{1}{6}\frac{1}{x^6} + C$

$y = \frac{1}{2}x^4 + \frac{1}{6}\frac{1}{x^4} + Cx^2$

at $x = -1$, $y = 2$

$2 = \frac{1}{2} + \frac{1}{6} + C$ so $C = 2 - \frac{1}{2} - \frac{1}{6}$
$= 2 - \frac{2}{3} = \frac{4}{3}$

$$\boxed{y = \frac{1}{2}x^4 + \frac{4}{3}x^2 + \frac{1}{6}x^{-4}}$$

(b) State the interval of solution.

since $P(x) = -\frac{2}{x}$, interval is either $x < 0$ or $x > 0$

from initial condition, $\boxed{x < 0}$

(c) Identify any transient terms in the solution.

the $\frac{1}{6}x^{-4}$ term is transient

3. (4 points) Solve the following DE.

$$\frac{dy}{dx} = -3 + \sqrt[3]{3x+y}$$

let $u = 3x + y$

$\frac{du}{dx} = 3 + \frac{dy}{dx}$ and $\frac{dy}{dx} = \frac{du}{dx} - 3$

DE becomes

$$\frac{du}{dx} - 3 = -3 + \sqrt[3]{u}$$

$$\frac{du}{dx} = u^{1/3}$$

$$\int \frac{du}{u^{1/3}} = \int dx$$

$$\frac{3}{2} u^{2/3} = x + C$$

$$\boxed{\frac{3}{2}(3x+y)^{2/3} = x + C}$$

4. (5 points) Consider the following differential equation.

$$\underbrace{(y^2 + xy^3)}_{M}\,dx + \underbrace{(y^3e^{-2y} - xy)}_{N}\,dy = 0$$

(a) Show that this DE is not exact in its current form.

(b) Find an appropriate integrating factor.

(c) Solve the DE.

a) $\frac{\partial M}{\partial y} = 2y + 3xy^2$, $\frac{\partial N}{\partial x} = -y$, since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, not exact

b) $\frac{M_y - N_x}{N} = \frac{2y + 3xy^2 + y}{y^3 e^{-2y} - xy}$ ← not a function of x only

$$\frac{N_x - M_y}{M} = \frac{-y - 2y - 3xy^2}{y^2 + xy^3}$$

$$= \frac{-3y - 3xy^2}{y^2 + xy^3} = \frac{-3y(1+xy)}{y^2(1+xy)} = \frac{-3}{y}$$

so $IF = e^{\int \frac{N_x - M_y}{M} dy} = e^{\int -\frac{3}{y} dy}$

$$= e^{-3\ln|y|}$$

$$= \frac{1}{|y|^3} = \frac{1}{y^3}$$

DE is now $\left(\frac{1}{y} + x\right)dx + \left(e^{-2y} - \frac{x}{y^2}\right)dy = 0$

note: $\frac{\partial M}{\partial y} = -\frac{1}{y^2}$

$\frac{\partial N}{\partial x} = -\frac{1}{y^2}$

though you don't need to show this

so $f(x,y) = C$

$$= \int M\,dx = \frac{x}{y} + \tfrac{1}{2}x^2 + g(y)$$

$$= \int N\,dy = -\tfrac{1}{2}e^{-2y} + \frac{x}{y} + h(x)$$

$$f(x,y) = \boxed{\frac{x}{y} + \tfrac{1}{2}x^2 - \tfrac{1}{2}e^{-2y} = C}$$

5. (6 points) When a webpage is moved or deleted, any links that point to the old location are said to have "link rot". Two computer scientists studying "link rot" found that the rate of change in the number of working links on a webpage is proportional to the number of currently working links on that page.

The researchers created a page with 510 links to science education resources in August 2000. Exactly two years later, they found that only 370 of those links were still working. If their model is accurate, how long will it take (from the original date of August 2000) for only half of the original links to still work?

let N = number of working links

$$\frac{dN}{dt} = kN$$

(and we expect k to be negative since N is decreasing)

$$\frac{dN}{N} = k\,dt$$

separable and linear 1st order in N

$$\ln|N| = kt + C$$

- also okay to drop absolute value since N = number of links is a non-negative integer

$$|N| = e^{kt+C}$$

$$N = \pm e^{kt} e^{C} = C_1$$

$$N = C_1 e^{kt}$$

but at $t=0$, $N = N_0$ so $N_0 = C_1 e^0$

and $\boxed{N = N_0 e^{kt}}$

so $N_0 = 510$ and at $t = 2$, $N = 370$

$$370 = 510 e^{k \cdot 2}$$

$$\frac{37}{51} = e^{2k}$$

$$\ln\left(\frac{37}{51}\right) = 2k$$

$$k = \frac{1}{2}\ln\left(\frac{37}{51}\right)$$

So $t_{1/2} = ?$

$$\frac{1}{2}\cancel{N_0} = \cancel{N_0} e^{kt}$$

$$\frac{1}{2} = e^{kt}$$

$$\ln\left(\tfrac{1}{2}\right) = kt$$

$$t = \frac{\ln(1/2)}{k} = \frac{2\ln(1/2)}{\ln(37/51)}$$

$$t \approx 4.3 \text{ years}$$

(4 years, 3 months, 26 days if you insist)